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Name : Sam Phere

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School / College : ______

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		Modern Theory of PDEs		

MA534 Modern Theory of PDEs

Jan 4, 2019

Prof. Mayukh Mukherjee

Lec 1

Office Hours: ~~Mon 10-11 am~~ Wednesday 4pm-5pm

Prerequisites
will be ~~rec~~ recalled

Extra problem sessions.
Saturday 10 am (Alternate)

Additional Books: 1. Gregory Eskin (Advanced)
Taylor → more geometric and topological
2. Kesavan (more fitted to the course)
3. Brezis
4. Hörmander (4 volumes) (The analysis book) very difficult.
Bible of PDEs.
*5. Jürgen ~~Jost~~ (Cognitive Science) typos ~~don't~~ trust.

some Background material:

1st HW

1) Real Analysis / Complex Analysis
Heine-Borel Thm → Ascoli-Arzela thm
Cauchy integral of holomorphic functions ~ Mean Value Theorem.

2) Functional / Measure / Fourier Analysis → $L^2 \to L^2$.
→ Hahn Banach thm. → L^p spaces. dual of L^p is L^q.

~~3)~~. Riesz Representation Thm.

3) Multivariable Calculus
Local Analysis. Mixed partial derivatives agree → Local analytic statements.
Stokes Thm: Global ~ Topological

Review Elliptic PDEs
[Laplacian. Elliptic Properties.

"Review": We will start with ~~the~~ Laplacian Δ, which is special enough so that many proofs are easier, but general enough ~~that~~ "many" facts generalise

$\Omega \to$ $\boxed{\text{open connected}}$ subset of $\mathbb{R}^n$
domain

$u : \Omega \to \mathbb{R}$ or $\mathbb{C}$

We will consider complex valued func^ns

<u>Laplacian</u> $\Delta = \partial^2_{x_1} + \partial^2_{x_2} + \dots \partial^2_{x_n}$

2nd order with rotatl terms

Harmonic Functions : $\Delta u = 0$

~~those that~~ satisfy

for all <u>dimensions</u>

Particularly, in dim $n = 2$, harmonic functions are real parts of holomorphic <u>functions</u>

(complex analytic)

$\Rightarrow$ Cauchy-Riemann

differentiable

$\Rightarrow$ can be expressed as series

Apriori: a harmonic function needs to be C^2

<u>Standard</u> : Rudin Real and Complex Analysis

$\to$ to make ~~def~~ defined $\frac{\partial^2}{\partial x^2}$

$C^k \to$ 1st k derivatives exist and last one is cont.

(In dim 3 no real part of "Complex")

$\to$ But it turns out to be C^∞. (Even more $\to$ real Analytic)

comes from Elliptic Regularity. (Proof <u>later</u>)

This is an inbuilt mechanism of elliptic operators

(Elliptic Regularity!)

What happens when $n = 3$?

<u>Properties of Harmonic functions</u> :

Let $x \in \underset{\text{domain}}{\Omega} \subseteq \mathbb{R}^n$, $u \in C^2(\Omega)$

u attains local max at x.

Then ~~is~~ $\nabla u|_x = 0$, $\Delta u|_x \leq 0$

Hess $u|_x$ second order $\to$ partial derivative

(Hessian matrix is -ve semi definite)

$\Delta = \text{tr}(\text{Hess})$

Eigenvalues add -ve

(Spivak : Multivar <u>Calculus</u>)

1) Variational characteristic of Harmonic functions

Harmonic functions are critical points of the Dirichlet energy : $E(v) = \int_\Omega |\nabla v|^2 \, dx$

E functional : v is function $\to$ number

suppose $\Delta u = 0$
u harmonic

change slightly $\to E(u+t\varphi)$

take $\left.\frac{d}{dt}\right|_{t=0} E(u+t\varphi) = 0$

real valued f : $\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} f(x+\varepsilon y) = 0 \iff x$ critical point of f.

minimize a functional $\xrightarrow{\text{variational Calculus}}$ the minimizer satisfies PDE.

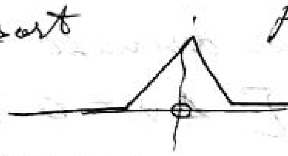
$\varphi \in C_c^\infty(\Omega)$
c : compact support.

Support of a function
$\text{supp } u = \{x \in \Omega \mid u(x) \neq 0\}$ $\to$ take all those points where $u(x) \neq 0$

support is compact $C_c^k(\Omega) = \{u \in C^k(\Omega)$

" doesn't reach boundary "

Find smooth func of compact support



Claim : $\left.\frac{d}{dt}\right|_{t=0} E(u+t\varphi) = 0$ if u harmonic

$E(u+t\varphi) = \int_\Omega |\nabla(u+t\varphi)|^2 \, dx$; diff is linear $\nabla(a+b) = \nabla a + \nabla b$

$= \int_\Omega |\nabla u|^2 + t^2 \int |\nabla \varphi|^2 + 2t \int \nabla\varphi . \nabla u$

t^2 term $\to$ doesn't survive ~~integr~~ differn

$\frac{dE}{dt} = 2 \int_\Omega \nabla\varphi . \nabla u$

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Gauss' Green Identity (jazzed up version of divergence thm)

Integr^ By parts

$$= -2 \int_{\Omega} \varphi \Delta u + 2 \int_{\partial\Omega} \varphi \frac{\partial u}{\partial n}$$

$\Delta u \to 0$; $\varphi \to 0$ at Boundary, C^∞

$= 0$

Converse also true

$$\frac{d}{dt}\Big|_{t=0} E(u + t\varphi) = -2 \int_{\Omega} \varphi \Delta u$$

critical pt : 0 for all φ

$$\int_{\Omega} \vartheta \varphi = 0 \text{ for all } \varphi \in C^\infty$$

$\Rightarrow \vartheta = 0$

$\therefore \Delta u = 0$

Proof by Contradic^ $\vartheta \neq 0 \to$ either +ve or −ve
choose Ω s.t. and choose $\varphi > 0$

Lec 2 Jan 8, 2019

Next Saturday HW1 Sat 10am volunteers Mayukh Mukherjee

Recalling imp concepts.

Office Hours: Wed 4–5pm.

Recall: Harmonic functions property

Harmonic $\to$ complex analysis $\to$ heat and wave $\to$ Distributions.

We discussed certain properties of harmonic functions on $\mathbb{R}^n$

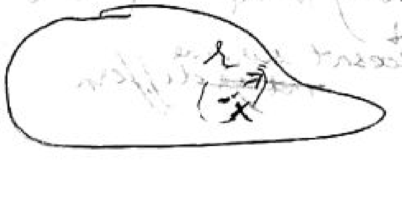
$[\Delta u = 0]$ $\quad \partial^2_{x_1} + \dots + \partial^2_{x_n}$

$u \in C^2(\mathbb{R}^n)$

Mean Value Theorem for Harmonic functions

Domain = open and connected

(Here) $\Omega \subseteq \mathbb{R}^n$



$u \in C^2(\Omega)$, $\Delta u = 0$

Mean Value Thm for Harmonic Func^n

$$u(x) = \frac{1}{|B_r(x)|}\int_{B_r(x)} u(y)\,dy = \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} u\,ds$$

Avg

Integrate : Multivar Calculus

If Boundary proved

Then $\int$ over vol to get

Proof : $\int_{B_r} \Delta u \overset{GGI}{=} \int_{\partial B_r} \frac{\partial u}{\partial n}\,ds$

Integr^n by parts: (Stokes, Gauss Green Identity
Building from div. theorem. GGI

$$\int_{B_r} \Delta u \cdot 1 \overset{GGI}{=} \int \nabla u \cancel{\nabla(1)}^{=0} + \int_{\partial B_r} 1\,\frac{\partial u}{\partial n}\,ds$$

$$\therefore \int_{B_r} \Delta u = \int_{\partial B_r} \frac{\partial u}{\partial n}\,ds$$

for u harmonic
$\nabla u = 0$

$$\therefore 0 = \int_{B_r(x)} \Delta u = \int_{\partial B_r} \frac{\partial u}{\partial n}\,ds$$

change $\underset{=}{}$ of variables $\quad r^{n-1}\int \frac{\partial}{\partial r} u(x+rw)\,dw$

bring centre to $\partial B_1(0)$

$r = 1$

Dominated convergence theory (Measure Theory) $\quad = \quad r^{n-1}\frac{\partial}{\partial r}\int_{\partial B_1(0)} u(x+rw)\,dw$

is

$-r\sin$

$r\cos$

$r\sin$

$$\lim_{n\to\infty} \int f_n \stackrel{?}{=} \int \lim_{n\to\infty} f_n$$

lim same as derivative)

monotone convergence theorem MCT → less applic (more testing

dominated convergence theorem DCT

("Bounded above L^1")

good measure theory

Real and Complex ← Rudin, Royden Real Analysis

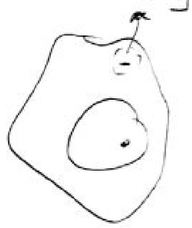
Evans's Appendix

$$0 = r^{n-1} \frac{\partial}{\partial r} \int_{\partial B_1(0)} u(x+rw)\, dw$$

↓ change of scale   ↓ change of origin

$$\frac{\partial}{\partial r} \int_{\partial B_1(0)} u(x+rw)\, dw = 0$$

r depends



$$\int_{\partial B_1(0)} u(x+rw)\, dw = \text{const} = c$$

↓ scale back

$$\int_{\partial B_1(0)} r^{n-1}\, u(x+rw)\, dw = c r^{n-1}$$

$\Downarrow$

$$\int_{\partial B_r(x)} u\, ds = c r^{n-1}$$

$$\frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u\, dS = \text{constant}.$$

u dominated by something $\in L^1$ (integrable)

Converse

Thm Let u be cont
$u \in C(\Omega)$ and u satisfies
EMVP

$$u(x) = \frac{1}{|B_r(x)|}\int_{B_r(x)} u\,dy = \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} u\,dS$$

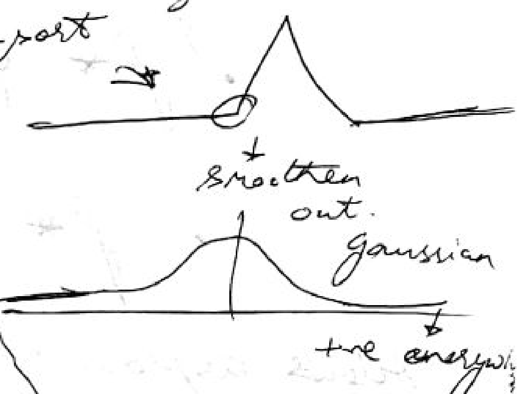
Elliptic PDE (Han - Lin)

~~then~~ u is harmonic and u is smooth
↓
all partial derivatives exist

Proof: (Mollifiers) ← (Evans Appendix)

Important to introduce
1st: smooth ~~func~~^{n} ~~with~~ compact support.
closure of set where non-zero

Continuous func^n of compact support →



Choose $\varphi \in C_c^\infty(B_1(0))$
↓
C_c^∞ : compact support
C_0 → also used somewhere
support not reaching boundary (open set)
extend by 0 → ~~on~~ rest of $\mathbb{R}^n$

such that

$$\int_{B_1(0)} \varphi = 1$$

Bump functions

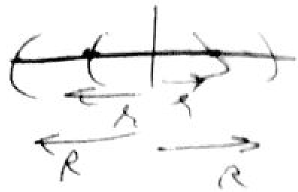
and φ is radially ~~sym~~ function.
(symmetry in $\theta, \phi, \ldots$)
only distance dependent

construct $\mathbb{R}^1$ ~~then~~ rotate to get $\mathbb{R}^n$

Define $\varphi_\varepsilon(z) = \frac{1}{\varepsilon^n}\varphi\left(\frac{z}{\varepsilon}\right)$

→ still smooth
smaller supports, integral 1.

def mollification



must be same symmetric

$\underbrace{f * \varphi_\xi}$ is the mollifier of f

$\to$ convolution

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy$$

(Idea later)

Properties A few nice properties

$f \in L^p$

- $f * \varphi_\varepsilon$ is smooth $\to$ all ~~partial~~ derivatives exist
- $f * \varphi_\varepsilon \xrightarrow[\text{a.e.}]{} f$ as $\varepsilon \to 0$ (proof details but no new idea)
  almost everywhere pointwise

Allows approxim^n of highly broken func^n ~~as~~ with smooth

Take closed, smooth func^n Weierstrass $\to$ Approx by polynomials

L^ps don't have to be naturally behaved

C_c is dense in L^p ~~for~~ $\mathbb{R}^n$ locally compact (Rudin)

Not only approximation exists,

it can be done by convolution of with

a ~~s~~ nice thing : φ_ξ

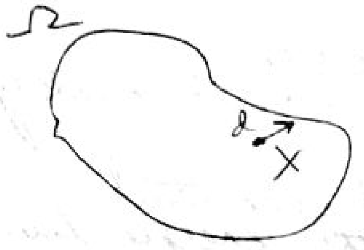
$$\partial_{x_i}(f * g) = (\partial_{x_i} f) * g = f * (\partial_{x_i} g)$$

here not smooth ×

Check! $\to$ pass derivative inside $\int$ (DCT)

Differentiating $f * g$

→ diff only one $\therefore$ g is smooth
$f * \partial_{x_i} g$ $\therefore$ we are done



$d =$ dist $(x, \partial\Omega)$

can't take Boundary to be C^1
(proof is actually local property so doesn't matter)

choose $\varepsilon < d$

choose $f = \varphi$ $g = u(y)$

$$u * \varphi_\varepsilon(x) = \int_\Omega u(y)\, \varphi_\varepsilon(x-y)\, dy$$

$x-y$ or $y-x$ evaluate same φ_ε (radial)

$$= \int_\Omega u(y)\, \varphi_\varepsilon(y-x)\, dy$$

$$\underset{y \to y+x}{\overset{\text{change of var}}{=}} \int_{\Omega = |y|<\varepsilon} u(x+y)\, \varphi_\varepsilon(y)\, dy$$

$$\overset{\text{def of } \varphi_\varepsilon}{=} \frac{1}{\varepsilon^n} \int_{|y|<\varepsilon} u(x+y)\, \varphi(y/\varepsilon)\, dy$$

$$\overset{\text{change of var}}{=} \int_{|y|<1} u(x+y\varepsilon)\, \varphi(y)\, dy$$

Convert to polar coord $y = (y_1, y_2, \dots, y_n)$

$\mathbb{R}^n$ vol $\left\{ dx_1 dx_2 \right.$
$\left. r^{n-1} dr\, d\omega \right.$ $\omega \in S^{n-1}$

$$\overset{\text{polar}}{=} \int_0^1 r^{n-1} \varphi(r) \int_{\partial B_1(0)} u(x + \varepsilon r \omega)\, d\omega\, dr$$

Jan 11 **Recall** Trying to prove continuity + MVP
$\Downarrow$
smoothness + harmonicity

Recap 1. Variational Characterisation of harmonic functions.

$\Delta u = 0$, $u \in C^2(\Omega) \cap C(\bar{\Omega})$ $\to$ cont boundary

$\Omega \subseteq \mathbb{R}^n$ open and connected

Harmonic func^ns are precisely critical points of the energy functional

$$E(u) = \int_\Omega |\nabla u|^2$$

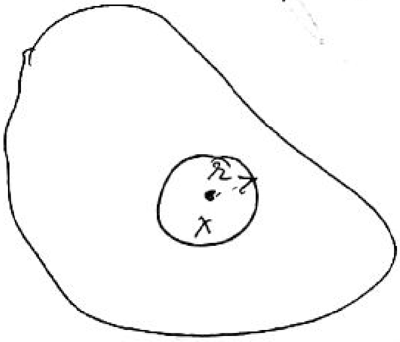
$\hookrightarrow$ will be made more formal later on.
(critical points on what functional space)

$E(u + t\varphi)$ $\varphi \in C_c^\infty(\Omega)$
perturb func^n slightly — smooth function of compact support.

$$\frac{d}{dt}\Big|_{t=0} E(u+t\varphi) = 0$$

There are ∞ dimension Banach spaces
where is the func^n coming from: $u \in H_0^1(\Omega)$ Sobolev spaces

2. Mean value property of harmonic func^n



$\Delta u = 0$ on Ω

$$u(x) = \frac{1}{|B_r(x)|}\int_{B_r(x)} u = \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} u$$

($|S|$ = volume of S)

3) converse also true.

Thm Let $u \in C(\Omega)$ have MVP

$$u(x) = \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} u$$

To prove: u is harmonic and u is smooth.

Proof Before proving, we introduce an important technical tool → Mollifiers

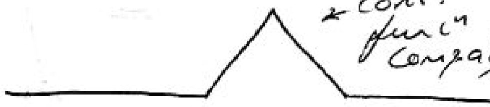
Take a radial bump function

φ s.t. $\operatorname{supp} \varphi \subset B_1(0)$

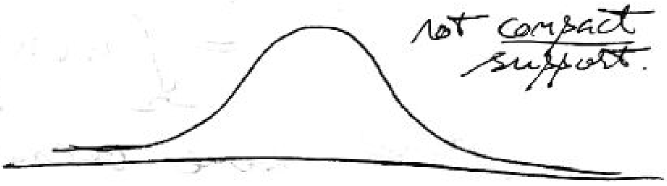
Bump function is a smooth function of compact support.

(Existence HW1 → Ref. Evans) (0 outside a bounded set)

Choose φ s.t. $\displaystyle\int_{B_1(0)} \varphi = 1$ (normalize)



Define $\varphi_\varepsilon(z) = \dfrac{1}{\varepsilon^n}\, \varphi(z/\varepsilon)$



so that automatically $\int \varphi_\varepsilon(z) = 1$

family of func → (smaller and smaller supports)

convolution of two functions f and g

→ (any book on Harmonic Analysis)

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy$$

substitute $t = x-y$ → $y = x-t$, $dy = -dt$ $\Big|\; = -\displaystyle\int_{-\mathbb{R}^n} f(t)\, g(x-t)\, dt = \int f(y)\, g(x-y)\, dy$

Useful property of convolutions

$$\partial_{x_j}(f*g) = (\partial_{x_j} f) * g = f * (\partial_{x_j} g)$$

~~mollification~~ mollifiers → $f \in L^p$, $\boxed{f * \varphi_\varepsilon} \xrightarrow{L^p} f$ almost everywhere (proof in Evans) (no new ideas in proof)

smooth

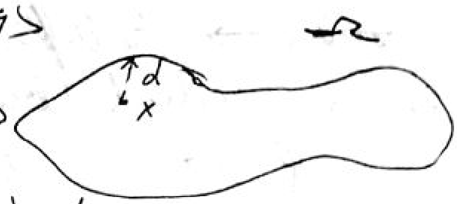
Approximate any L^p func by smooth func.

f: broken discontinuous things

Proof MVP $\rightarrow$ smooth and harmonic. (local property)

Har and Lin

choose $\varepsilon < d = \text{dist}(x, \partial\Omega)$



u is cont, satisfying MVP

$$\underbrace{u * \varphi_\varepsilon(x)}_{\text{is smooth}} = \int_\Omega u(y)\,\varphi_\varepsilon(x-y)\,dy$$

$$= \int_\Omega u(y)\,\varphi_\varepsilon(y-x)\,dy$$

φ_ε is radial func^n same value at $\vec{x}$ and $-\vec{x}$

change of var $y - x = y'$ and relabel $y' = y$

$$= \int_\Omega u(x+y)\,\varphi_\varepsilon(y)\,dy$$

(domain same because support only in ε ball.) 0 everywhere else.

$$\overset{\text{def}}{\underset{\text{of } \varphi_\varepsilon}{=}} \frac{1}{\varepsilon^n} \int_{|y|<\varepsilon} u(x+y)\,\varphi(y/\varepsilon)\,dy$$

supp $\varphi_\varepsilon \in B_\varepsilon(0)$

Change of var:

$\frac{y}{\varepsilon} = y' \rightarrow$ limits $|y| < 1$

y is in n dimen^n scaled ε^n

$y = \varepsilon y'$

$dy = \varepsilon^n dy'$

$$= \frac{\cancel{\varepsilon^n}}{\cancel{\varepsilon^n}} \int_{|y|<1} u(x+y\varepsilon)\,\varphi(y)\,dy$$

$y \leftrightarrow$ polar coordinates (write only in radial $\varphi(y)$)

$x_1\, x_2\, x_3 \ldots x_n \rightarrow r, \omega$

$dx_1\, dx_2 \ldots dx_n$ $\quad r \in (0, \infty)$ $\quad \omega \in S^{n-1}$ $(n-1)$

$\quad = r^{n-1}\,dr\,d\omega$

radial can be pulled out.

$$\underset{\text{polar}}{=} \int_0^1 r^{n-1}\varphi(r) \left(\int_{\partial B_1(0)} u(x+\varepsilon r\omega)\,d\omega \right) dr$$

$|y| \le 1$ $r = 0$ to 1

shift coordinates

$$\int_0^1 r^{n-1}\varphi(r) \left(\int_{\partial B_1(x)} u(\varepsilon r\omega)\,d\omega \right) dr$$

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$$= \int_0^1 r^{n-1}\varphi(r)\left(\int_{\partial B_{\varepsilon r}(x)} (\varepsilon r)^{1-n} u(\theta)\,d\theta\right) dr$$

$\varepsilon r w_i = \theta_i$

$(\varepsilon r)^{n-1} dw = d\theta$

$$= \int_0^1 \varphi(r)\,\varepsilon^{1-n}\left[\int_{\partial B_{\varepsilon r}(x)} u(\theta)\,d\theta\right] dr$$

MVP

$$= \int_0^1 \varphi(r)\,\varepsilon^{1-n}(\varepsilon r)^{n-1}\underbrace{|S^{n-1}|}_{(r=1)\text{ sphere}}\,u(x)\,dr$$

$$= u(x)\int_0^1 \cancel{r^{n-1}}\,\varphi(r)\,r^{n-1}|S^{n-1}|\,dr$$

$$= u(x)\underbrace{\int_0^1\int_{S^{n-1}} \varphi(r)\,r^{n-1}\,dr\,dw}_{\text{integral of } \varphi(r) \text{ over } B_1(0) = 1 \text{ By def}^n}$$

So u is smooth

Exercise: Prove u is harmonic

Hint: Go through our proof of MVT

Corollary: Harmonic functions are smooth.
(specific case of elliptic regularity)

Proof: harmonic func^{n}s are C^2 + MVP $\Rightarrow$ smooth.

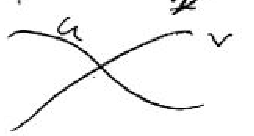
smooth: all partial deriv exist

real analytic $\to$ taylor series converges

Find func^n ~~that~~ is smooth but not real analytic

real analytic: restrict points where f can vanishes only on discrete points

0 set of harmonic ~~analytic~~: only at lines
Cant have limit point.



Cauchy Riemann

~~2~~ not possi

Recall : $\Omega \subseteq \mathbb{R}^n$

u is cont + MVP

$\Downarrow$

u is C^∞ and harmonic

cont

$$\int_{B_r(x)} \Delta u = r^{n-1} \frac{\partial}{\partial r} \int_{\partial B_1(0)} u(x + r\omega)\, d\omega$$

$\int \int$ I + dominant convergence

$$= r^{n-1} \frac{\partial}{\partial r} \left[u(x) |S^{n-1}| \right]$$

↳ does not depend on r

$$= 0$$

Claim: If $\int_{B_r(x)} \Delta u = 0 \quad \forall x \in \Omega, \ \forall r < \mathrm{dist}(x, \partial\Omega)$

$\Downarrow$

$\Delta u = 0$

u is C^2 on Ω and continuous on boundary

If +ve somewhere integrate over the neighbourhood → continuous ∴ +ve

→ get $\int \Delta u > 0$ ~~But~~ contradic^n

Corollary. Uniqueness of dirichlet problem:

$\Omega \subseteq \mathbb{R}^n$ $f \in C^0(\partial\Omega)$

Find a harmonic extension of f to Ω.

ie

solve $(*)$ $\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = f \end{cases}$

Claim. Sol^n of dirichlet problem if $\boxed{\text{it exists}}$ is unique

→ later Perron method and generaliz^{ns}.

Proof. If not unique, let u, v solve $(*)$

$w = u - v \qquad \Delta w = 0$

$w|_{\partial\Omega} = 0$

Let's say $w \neq 0$ ~~at this~~ somewhere

Idea

want to prove that

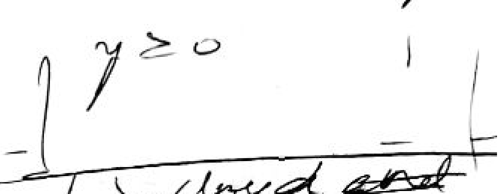
$\max_{\bar\Omega}$ of w and $\min_{\bar\Omega}$ of w

occurs on ~~the~~ boundary $\partial\Omega$

To prove max on $\partial\Omega$

suppose not:

max is somewhere inside:

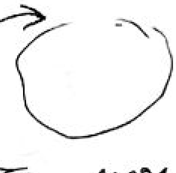


closed ~~and~~ domain.

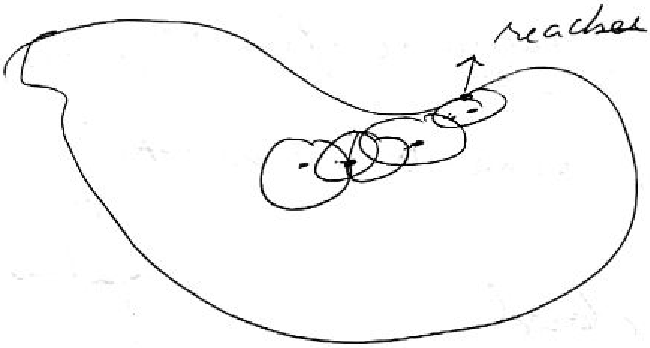
This is the Boundary

is max

But also avg of



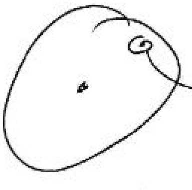
reaches Boundary.



avg = max

only if

max = every element.

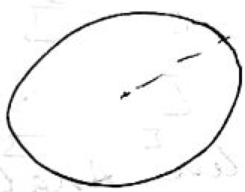


suppose all values not equal

Choose point where ~~f(x)~~ $u(x) <$ max.

Then ~~there~~ must be

Remark: 2D



max of (Holomorphic f) $|f|$ is attained on ∂B

Maximum modulus principle.

Special case: $d=2$ $\Omega = B$

In first complex analysis course:

Liouville's Thm: (for holomorphic func^s $f: \mathbb{C} \to \mathbb{R}$)

A bounded holomorphic f is constant

↳ (that are entire: domain of def^n all of $\mathbb{C}$

Comment: Only maximum principle is not enough to prove Liouville.

i.e. one can find func^s $f: \mathbb{C} \to \mathbb{C}$ st f is bounded, f satisfies max principle, f is non constant.

(These are not entire otherwise Liouville fails)

↳ i.e. holomorphic on all $\mathbb{C}$

ex: $f(z) = \dfrac{z}{1+|z|}$

Liouville's thm: Global — Look at boundedness over domain
max modulus: — max attained in neighbourhood ball

No Holomorphic func^n can take only real values.

HW Prove (Holomorphic func^{n}s ~~take~~ are open maps.
$\mathbb{C} \to \mathbb{C}$

Hint: Follows from maximum principle.

Actually: in general: also true for harmonic func^{n}s $\mathbb{R}^n \to \mathbb{R}$

If image ~~map~~ is not open ~~then~~ some bound on the

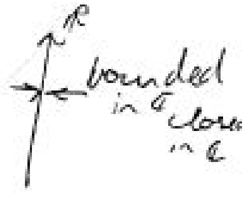


Image of open set $\to$ interval.

1st

interior point ~~takes~~ max.

Proof in Rudin's Real and Complex Analysis.

Corollary: There are no non-constant real valued holomorphic functions $\mathbb{C} \to \mathbb{C}$

Real line is not open in $\mathbb{C}$

Hw. Cauchy Goursat $\underline{th}$ + not same as MVT

$$\int f = \int u + i \int v \qquad \int u = u(x_0) \quad \int v = v(x_0)$$

$$0 = u(x_0) + i v(x_0)$$

See the diff in change of variables

Pointwise gradient estimates

$u \in C^2(B_R(x_0)) \cap C(\overline{B_R(x_0)})$

$u \geq 0$, $\Delta u = 0$ — depends only on n

Then, $|\nabla u(x_0)| \leq \frac{C(n)}{R} u(x_0)$

Proof: $\partial_{x_i} u$ is harmonic $\forall i$

$\Delta(\partial_{x_i} u) = 0$ — Δ commutes because $u \in C^3$
$= \partial_{x_i}(\Delta u) = 0$

~~$\Delta \partial_{x_i} =$~~

apply MVT on $\partial_{x_i} u(x_0)$

$$\partial_{x_i} u(x_0) \overset{MVT}{=} \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} \partial_{x_i}(u)$$

integrate by parts

$$\left| \frac{1}{|B_R|} \int_{\partial B_R} u \nu_i \right| \qquad \nu = \nu_1, \dots, \nu_n \text{ outward pointing unit normal to } \partial\Omega.$$

$|\nabla u| = \left(\sum (\partial_{x_i} u)^2\right)^{1/2}$

every ∂_{x_i} bounded by $\frac{C(n) u(x_0)}{R}$

$$\leq \frac{1}{|B_R|} \int_{\partial B_R} |u| \overset{u \geq 0}{=} \frac{1}{|B_R|} \int_{\partial B_R} u$$

$$\leq \frac{1}{|B_R|} |\partial B_R| \, u(x_0)$$

$$= \frac{C(n)}{R} u(x_0)$$

$$C(n) \to \frac{|S^{n-1}|}{|\mathrm{Vol}\, S^n|}$$

Corollary: Liouville: Bounded harmonic functions $f: \mathbb{R}^n \to \mathbb{R}$ are constant.

wlog $u \geq 0$, $\Delta u = 0$
Bounded. Thus shift higher by min $f(x) = f(x) + \min f(x)$

$$|\nabla u(x)| \leq \frac{C(n)}{R} u(x)$$

If harmonic everywhere. take $R \to \infty$ $|\nabla u(x)| \leq 0$

$\to u(x) = \text{const}$ $\boxed{\nabla u(x) = 0}$

(Jan 25, Friday)

Monday → 11 → 12:30

Presentation topics : ↳ Perron's method (1934) : Existence of Dirichlet Problems.

Given a function $f \in C(\partial\Omega)$ → $\Omega \subseteq \mathbb{R}^n$ bounded domain.

can we solve

$$\Delta u = 0 \quad u|_{\partial\Omega} = f \Big\} * \text{ harmonic extension.}$$

Heuristic : $\partial\Omega$ is "nice enough" ⇔ can solve (*)

Boundary points should be regular. Probabilistic interpretation.

⇒ Hölder regularity of $\Delta u = f$
Folland Thm 2.28

In next homework: counter example of $\left.\begin{array}{l}\Delta u \in C(\mathbb{R}^n) \\ \Downarrow \\ u \in C^2(\mathbb{R}^n)\end{array}\right\}$ False!

Later, Sobolev regularity

$Lu \in H^s \Rightarrow u \in H^{s+2m}$
↳ elliptic of order $2m$.

Hölder spaces $C^{k,\alpha}$

$$\Delta u \in C^{k,\alpha}(\mathbb{R}^n)$$
$$\Downarrow$$
$$u \in C^{k+2,\alpha}(\mathbb{R}^n)$$

Last class: Harnack $\Delta u = 0 \quad u \geq 0$

$$\sup_{\Omega} u \leq C(\Omega) \inf_{\Omega} u$$

Let's say bounded below.

$u \geq -100 \qquad u + 100 \geq 0$ is harmonic $\qquad \Delta \varphi = 0$

If 0 is attained ~~chain~~ take balls around everywhere 0

$v(x_m) \to 0 \quad x_m \to 0$

$\varepsilon > 0$ choose a ball

Favourite proof of Liouville :

MVT,

$$u(x) = \frac{1}{|B_r|}\int_{B_r(x)} u$$

$$u(y) = \frac{1}{|B_r|}\int_{B_r(y)} u$$

$$|\cancel{u(x)-u(y)}| \leq \frac{1}{|B_r|}\int_{B_r(x)\,\Delta\, B_r(y)} |u|$$

Recall: Proved pointwise gradient estimates for harmonic func^n $\Rightarrow$ Liouville Thm.

$f: \mathbb{C} \to B_1(0)$, $f(z) = \frac{z}{1+|z|}$ violates ↑ satisfies MVP
in $\mathbb{C}$, not holom.

Liouville: global

Applied ~~Liouville~~ MVT on derivative of harmonic func^n because derivative of harmonic func^n → again harmonic.

$B_r \subseteq \overline{B_r} \subseteq B_R$

Propos^n ~~$f: \mathbb{C} \to B_1(0) \subseteq \mathbb{C}$~~ $u \in C(\overline{B_R})$, $\Delta u = 0$

Inductively: Then $|\partial^\alpha u(x)| \le \frac{n^m e^{m-1} m!}{R^m} \max_{B_R} |u|$

$|\alpha| = m$ $\boxed{\partial_i = \frac{1}{i}\partial x_j}$ → For notational convenience to work with Fourier Transform

$\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}$ → $\alpha = \alpha_1+\alpha_2+\dots+\alpha_n$

fourier transf ← $\widehat{\partial_i f}(\xi) = \xi_i \hat{f}$

Let's say $x \xrightarrow{F} \xi$

$\widehat{\partial_j f}(\xi) = \xi_j \hat{f}(\xi)$

$\hat{f}(\xi) = \int f(x) e^{-ix\xi} dx$

multiply by ξ and push inside.

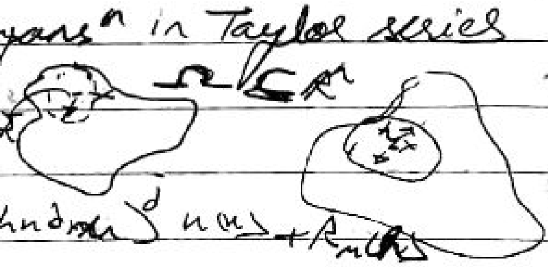
$\widehat{x_j f} = \partial_j \hat{f}$

$\Delta u = 0$
$|\xi|^2 \hat{u} = 0$

Proof: Linear PDE derivatives commute
Observe ~~that~~ $\partial^\alpha u$ is harmonic $\forall$ multiindex α.
Apply MVT on $\partial^\alpha u$.

Recall ~~that~~ real analytic $\Leftrightarrow$ local expans^n in Taylor series

Thm suppose $\Delta u = 0$ $u \in C(\Omega)$ $\Omega \subseteq \mathbb{R}^n$
and $h \in \mathbb{R}^n$ s.t. ~~$|h| < R$~~ s.t. $|h| < R$



$$u(x+h) = u(x) + \sum_{j=1}^{m-1} \frac{1}{j!}(h_1\partial x_1 + \dots + h_n \partial x_n)^j u(x) + R_m(h)$$

MVT: $R(h) = \frac{1}{m!}\left[(h_1\partial x_1 + \dots + h_n \partial x_n)^m\right] u(x+\theta h)$ $\quad 0 \le \theta \le 1$

Claim: $R(h) \longrightarrow 0$ as $m \to \infty$

$$|R_m(h)| \le \frac{1}{m!} \sum_{|\alpha|=m} |h^\alpha \partial^\alpha u(x+\theta h)|$$

$$\le \frac{1}{m!} \sum_{|\alpha|=m} \frac{n^m e^{m-1}}{R^m} m! \, |h|^{|\alpha|} \max_{B_{2R}} |u|$$

$\hookrightarrow$ sum gives 2^m terms

$|h^\alpha| = |h_1^{\alpha_1} h_2^{\alpha_2} \cdots h_n^{\alpha_n}| \le |h|^{\alpha_1} |h|^{\alpha_2} \cdots |h|^{\alpha_n} \le |h|^{m = \alpha}$

$$\le 2^m \frac{n^m e^{m-1}}{R^m} \max_{(B_R)} |u| \, |h|^m$$

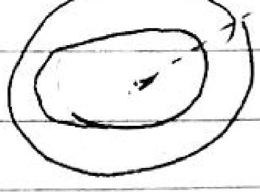
$$< \left(\frac{2ne|h|}{R}\right)^m \max |u|$$

Choose $|h| << \frac{R}{2ne} (\max_{B_R} |u|)^{1/m}$

Derivative of harmonic func^n $\to$ harmonic $\to$ get estimate on multiindex derivative $\to$ Taylor expand & remainder $\to 0$

Temp decay on curvature. ~~D~~ Liouville fails on connected R^n space

Harnack Inequality



Suppose $u \in C(\overline{B_{2R}(0)}) \subset R^n$

$u \ge 0$, $\Delta u = 0$

There exists a $C(n)$ s.t.

$$\sup_{B_R(0)} u \le C(n) \inf_{B_R(0)} u$$

$$C_1(n) \le \frac{u(x)}{u(y)} \le C_2(n)$$

Remark Harnack $\Longrightarrow$ Liouville.

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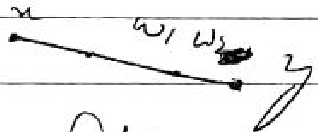
Proof: Trisect x, y :



$$u(w) \overset{MVT}{=} \frac{1}{|B(w,R/3)|} \int_{B(w,\frac{R}{3})} u \qquad ⨍ u$$

Notational Remark: $\frac{1}{|B(x,R)|} \int_{B(x,R)} u = ⨍_{B(x,R)} u$

$$u(w) = \frac{3^n}{|B_R(x)|} \int_{B_{R/3}(w)} u$$

Check: $B_{R/3}(w) \subseteq B_R(x)$

$u \geq 0$

Integral greater $\leq \frac{3^n}{|B_R(x)|} \int_{B_R(x)} u \overset{MVT}{=} 3^n u(x)$

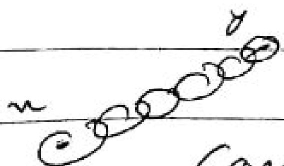
Inductively: $u(z) \leq 3^n u(w)$ and $u(y) \leq 3^n u(z)$

$u(y) \leq 3^{3n} u(x)$

also $u(x) \leq 3^{3n} u(y)$ $\Big\} \to 1 \leq 3^{6n}$ true.

HW: Extend Harnack to the following setting:
Suppose u is harmonic in Ω, $u \in C(\bar{\Omega})$
For any compact $K \subseteq \Omega$, $\exists\, C(\Omega, K) > 0$ s.t. if $u \geq 0$ in Ω, then

$$\frac{1}{C} u(y) \leq u(x) \leq C u(y) \quad \forall x, y \in K$$



Compact: any open cover has a finite subcover

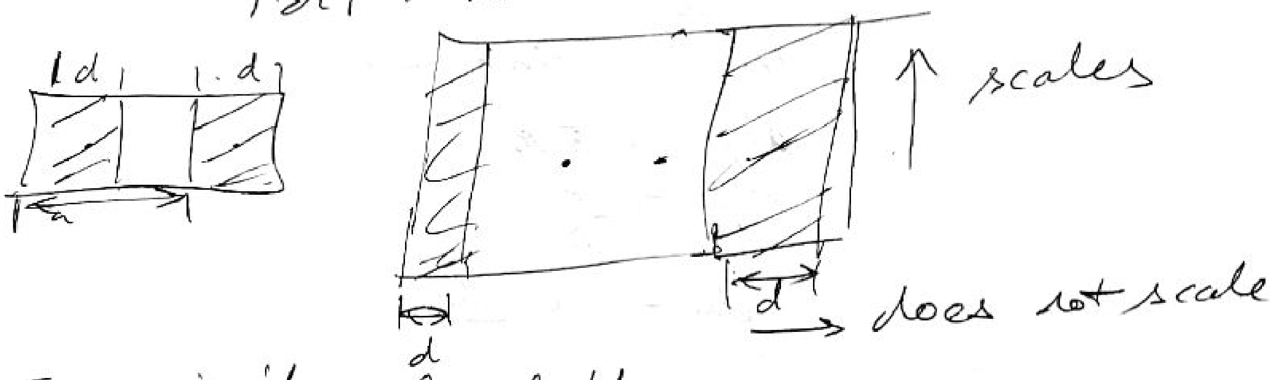
Suppose u is Bounded

$$|u(x)-u(y)| \le \frac{M}{|B_r(x)|}\,|B_r(x)\,\Delta\, B_r(y)|$$

claim: $|B_r(x)\,\Delta\, B_r(y)| \sim r^{n-1}$ whereas

↳ symmetric difference

$|B_r| \sim r^n$



Try similar for ball.

$$|u(x)-u(y)| \le \frac{M\, r^{n-1}}{r^n} = \frac{M}{r}$$

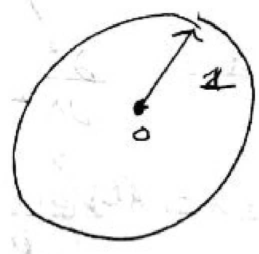
Choose $r \to \infty$ so that $|u(x)-u(y)| < \varepsilon$

$u(x) = u(y)$ $\forall x, y$.

Dirichlet Problem for $B_1(0) \subseteq \mathbb{R}^2$

$f \in C^1(S^1)$ (continuous on S^1)

$$\left.\begin{array}{l}\Delta u = 0\\ u|_S = f\end{array}\right\} \text{want to}$$



Important fact: $\underset{\partial_x^2+\partial_y^2}{\Delta} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$

$x = r\cos\theta$
$y = r\sin\theta$

$$\frac{\partial x}{\partial r} = J = \begin{bmatrix}\cos\theta & \sin\theta\\ -r\sin\theta & r\cos\theta\end{bmatrix}$$

Note: If u is radial, then $\Delta u = 0$ reduces to $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = 0$

ODE

(Find all radial harmonic ~~questions~~ sol^n in $\mathbb{R}^n$)

Solve ODE

In $\mathbb{R}^2$ $\quad u = a + b\log r$

$\Rightarrow \quad \frac{du}{dr} = v$

$$\frac{dv}{dr} + \frac{1}{r}v = 0$$

$$\frac{dv}{dr} = -\frac{v}{r}$$

$$\ln(v) = -\ln r + c$$

$$v = c\left(\frac{1}{r}\right)$$

$$\frac{du}{dr} = \frac{c}{r}$$

$$\int du = c\int\left(\frac{dr}{r}\right)$$

$$\boxed{u = c\ln(r) + d}$$

$$\boxed{u = a + b\log r}$$

separation of variables: Let $u(r,\theta) = g(r)\,h(\theta)$

(Uniqueness by maximum principle)
$\rightarrow$ No interior min/max $\Leftarrow$ MVT

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial \theta^2}\right)$$

$$\Delta u = h(\theta)\frac{d^2 g(r)}{dr^2} + \frac{h(\theta)}{r}\frac{dg(r)}{dr} + \frac{g(r)}{r^2}\frac{d^2 h(\theta)}{d\theta^2}$$

f is a 2π-periodic function on $\mathbb{R}$. So it has a Fourier expansion: $\quad f(\theta) = \sum_{k=0}^{\infty}(c_k\cos(k\theta) + d_k\sin k\theta)$

May not converge uniformly

Series: $x_1 + x_2 + x_3 + \ldots$
$+ x$

Cesaro $\quad s_N' = \frac{s_1 + s_2 + \ldots\ s_N}{N} \rightarrow x$

(partial sums)

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$\Delta u = 0 \Rightarrow \underbrace{\frac{r^2}{g}\frac{d^2g}{dr^2} + \frac{r}{g}\frac{dg}{dr}}_{r} + \underbrace{\frac{1}{h}\frac{d^2h}{d\theta^2}}_{\theta} = 0$

$\Rightarrow \frac{r^2}{g}\frac{d^2g}{dr^2} + \frac{r}{g}\frac{dg}{dr} = -c$ (1)

$\frac{1}{h}\frac{d^2h}{d\theta^2} = +c$ (2)

$\hookrightarrow h'' = +ch$

$\therefore h = \begin{cases} a e^{\sqrt{c}\theta} + b e^{-\sqrt{c}\theta} & c > 0 \\ a + b\theta & c = 0 \\ a\cos(\sqrt{-c}\,\theta) + b\sin(\sqrt{-c}\,\theta) & c < 0 \end{cases}$

we want h to be 2π-periodic in θ

$c > 0$ not possible.

Also $c = 0 \Rightarrow b = 0$

$h(\theta) = \begin{cases} a & c = 0 \\ a\cos(\sqrt{-c})\theta + b\sin(\sqrt{-c}\,\theta), & c < 0 \end{cases}$

for $c = 0$ remaining (1)

$\frac{r^2}{g} g'' + \frac{r}{g} g' = 0$

$\boxed{g = a_0 + a_1 \log r}$

for $c < 0$ > for h to be 2π periodic, $c = -k^2$

$\frac{r^2}{g} g'' + \frac{r}{g} g'$ ~~k~~

$\sin(k\theta)$ is 2π periodic for $k \in$ int

$\sqrt{-c} = \text{int.}$

$-c = k^2$

$\boxed{c = -k^2}$

$\therefore$ (1) : $\boxed{\frac{r^2}{g} g'' + \frac{r}{g} g' \cancel{+0} = k^2}$

guess $g = r^m$

$\cancel{r^2}\ m(m-1)\cancel{r^{m-2}} + \cancel{r}\ m(\cancel{r^{m-1}}) = \cancel{g}\ k^2$

$m(m-1) + m = k^2$

$m^2 - \cancel{m} + \cancel{m} = k^2$

$\boxed{m = \pm k}$

$\boxed{g(r) = a_0 r^k + a_1 r^{-k}}$

~~Want~~ to avoid singularity ~~at origin~~

$u(r,\theta) = \sum_{k=1}^{\infty} (a_k r^k + b_k r^{-k})(a\cos(k\theta) + b\sin(k\theta))$

$+ (d_0 + e\log r) \times (\text{const})$

To avoid singularity at $\underline{\underline{r=0}}$. $\therefore b_n = 0,\ e = 0$

$u(r,\theta) = \sum_{k=0}^{\infty} \cancel{a_k} r^k (a_k\cos k\theta + b\sin k\theta)$

Final: $u(r,\theta) = \sum_{k=0}^{\infty} r^k (c_k\cos k\theta + d_k \sin k\theta)$

Let $A_N\ f = \sum_{k=0}^{N} r^k (c_k\cos k\theta + d_k\sin k\theta)$

$= c_0 + \sum_{k=1}^{N} r^k \left[\frac{1}{\pi}\int_{-\pi}^{\pi} f(\phi)\cos(k\phi)\,d\phi\ \cos k\theta\right.$

$\left. + \frac{1}{\pi}\int_{-\pi}^{\pi} f(\phi)\sin(k\phi)\,d\phi\ \sin k\theta\right]$

($\because$ Fourier ~~coeff~~ given by orthonormal: $c_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx$

series can be summed

$\underset{N\to\infty}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(\phi)(1-r^2)}{1+r^2-2r\cos(\theta-\phi)}\,d\phi$

Stein-Sha

Fourier Analysis

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This is also a convolution:

$$u(r,\theta) = P_r(\theta) * f(\theta)$$

$$\boxed{P_r(\theta) = \frac{1}{2\pi}\frac{(1-r^2)}{1+r^2-2r\cos\theta}}$$

check if: u is harmonic (Exercise: $u(r,\theta)$ is harmonic)

: $u|_{\partial\Omega} = f$

Wed $u(r,\theta) = P_r(\theta) * f(\theta)$ $\to$ f cont on S^1

$\Delta u = 0$ $\quad u|_{S^1} = f$

$u(r,\theta) \to f(\theta_0)$ as $r \to 1$, $\theta \to \theta_0$

$r \nearrow 1$

for $0 \le r < 1$

$P_r(\theta)$ satisfies $\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-r^2}{1+r^2-2r\cos\theta}\,d\theta = 1$

[Proof follows from MVT applied to $P_r(\theta)$]

$$\int_{-\pi}^{\pi} f*g = \int_{x-\pi}^{x+\pi} g*f \quad (2\pi \text{ periodicity})$$

$(f*g = g*f$ for radial symmetry?$)$ $\quad \int f(x-y)g(y)\,dy$ $\quad x-y = t$, $-dy = dt$

$\int f(y)\,g(x-y)\,dy$

$$u(r,\theta) = P_r(\theta) * f(\theta)$$
$$= f(\theta) * P_r(\theta)$$

$f(\theta_0)\times 1 = f(\theta_0)\int_{-\pi}^{\pi} P_r(\theta)\,d\theta$

$$u(r,\theta) - f(\theta_0) = \int_{-\pi}^{\pi} f(\theta-\phi)\,P_r(\phi)\,d\phi - \int f(\theta_0)P_r(\phi)\,d\phi$$

P_r is always +ve:

$$|u(r,\theta) - f(\theta_0)| \le \int_{-\pi}^{\pi} |f(\theta-\phi) - f(\theta)|\,P_r(\phi)\,d\phi$$

$$+\int_{-\pi}^{\pi} |f(\theta) - f(\theta_0)|\,P_r(\phi)\,d\phi$$

can be made $< \frac{\varepsilon}{3}$ by cont. of f

(P_r is bounded)

Jan 30 Wednesday

Prof Mayukh Mukherjee

Again partition:

$$\int_{|\phi|<\delta} |f(\theta-\phi) - f(\theta)|\, P_r(\phi)\, d\phi + \int_{|\phi|\geq\delta} |f(\theta-\phi) - f(\theta)|\, P_r(\phi)\, d\phi$$

All we have used is $\theta \to \theta_0$

Now use $r \to 1^-$

f is bounded : $|f| \leq M$

$$\int_{|\phi|\geq\delta} |f(\theta-\phi) - f(\theta)|\, P_r(\phi)\, d\phi$$

$$\leq 2M \int_{|\phi|\geq\delta} P_r(\phi)\, d\phi$$

When $\phi \to 0$, $\cos\phi \to 1$: $\dfrac{1-r^2}{1+r^2-2r\cos\phi} = \dfrac{1+r}{(1-r)}$

$\phi\,[\pi, \pi]$

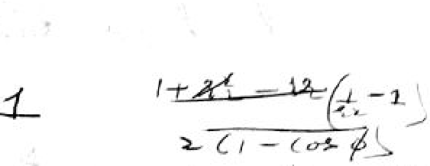
$\cos\phi \to$ ~~0~~ -1 to 1, to -1

$\therefore$ for $|\phi| \geq \delta$ $P_r(\phi) \to 0$ for $r \to 1$

$\dfrac{1+r^2-2r}{2(1-\cos\phi)} \left(\frac{1}{r}-1\right)$ non zero

Perron method: Jurgen Jost (PDEs Chap 3) New Edition.

(super harmonic) $\Delta u \leq 0$



Poincaré Inequality (Dirichlet version)

suppose $u \in C^1(\Omega)$, $u|_{\partial\Omega} = 0$

Ω bdd domain

Then: $\int_\Omega u^2 \leq \delta(\Omega) \int_\Omega |\nabla u|^2$ ← constant dependent on the domain

also works for C^1 closure

$$\|u\|^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2$$

Remark $\Delta u = 0$ is just the kernel of operator Δ

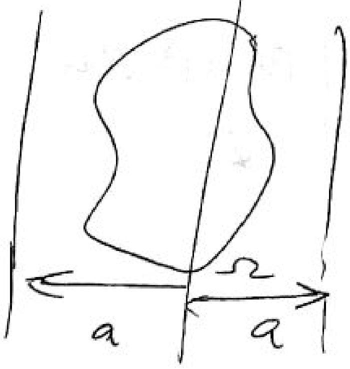
suppose φ is an eigenfunc^n

$$\int |\nabla \varphi|^2 \overset{GI}{=} \int_\Omega -\Delta\varphi\, \varphi = \lambda \int_\Omega \varphi^2$$

$$\frac{\int |\nabla \varphi|^2}{\int \varphi^2} = \lambda \rightarrow \text{spectrum of operator } \Delta$$

Later: $\delta(\Omega)$ is the lowest eigenvalue of the ~~the~~ domain Ω

Proof:



Bounded: Contained b/w two hyperplanes

Choose coordinates $x_1, x_2, \ldots x_n$ in such a way: $x_1 \rightarrow$ perpendicular to H

$x_2, \ldots x_n \rightarrow$ form hyperplane parallel to the ones before

$x \equiv (x_1, x^*)$ $\quad x^* = (x_2, \ldots x_n)$

$$u^2(x_1, x^*) \underset{\text{Thm of Calculus}}{\overset{\text{Fundam}}{=}} \int_{-a}^{x_1} \partial_{x_1} u^2 \, dx_1 \qquad \text{(Boundary term is 0)}$$

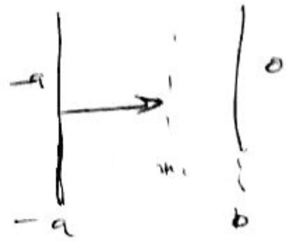
$$= 2\int_{-a}^{x_1} u\, \partial_{x_1} u \, dx_1$$

$$\underset{\text{Holder Ineq}}{\leq} 2 \left(\int_{-a}^{x_1} u^2\right)^{1/2} \left(\int_{-a}^{x_1} (\partial_{x_1} u)^2\right)^{1/2}$$

(1 term of the gradient $\leq |\nabla u|^2$)

~~C^1 under norm $\|u\|^2$~~ $\cancel{\Leftrightarrow \left(\int_{-a}^{0} u\, dx_1\right)^{1/2}}$

For $x_1 < 0$



$$\le 2\left(\int_{-a}^{0} u^2\, dx_1\right)^{1/2}\left(\int_{-a}^{0} (\partial_{x_1} u)^2\, dx_1\right)^{1/2}$$

RHS independent of x_1 $\qquad$ const $\times \int_{-a}^{0} dx_1 = a \times$ const

$$\underbrace{\int_{-a}^{0} u^2 (x_1, x^*)\, dx_1 \le 2a}_{\text{same}} \left(\int_{-a}^{0} u^2\, dx_1\right)^{1/2}\left(\int_{-a}^{0} (\partial_{x_1} u)^2\, dx_1\right)^{1/2}$$

$$\Rightarrow \int_{-a}^{0} u^2\, dx_1 \le 4a^2 \int_{-a}^{0} |\nabla u|^2\, dx_1$$

Similarly, one can show that

$$\int_{0}^{a} u^2\, dx_1 \le 4a^2 \int_{0}^{a} |\nabla u|^2\, dx_1$$

Adding, $\underbrace{\int_{-a}^{a} u^2\, dx_1}$ $\le 4a^2 \int_{-a}^{a} |\nabla u|^2\, dx_1$

This is a func^n of $\longrightarrow$ " $\qquad$ "
$(x_2, x_3 \ldots x_n)$

In other directions $f \le c g$

Then $\int f\, dx_2 dx_3 \ldots dx_n \le c \int g\, dx_2 dx_3 \ldots dx_n$

L_2 norm along all direc^n $\int_{R^n}$ is $\le$ $\int_{R^n}$

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Feb 1 2019

HW 2 uploaded
HW1: Wednesday 4:5 Bergman Kernel
2 thms from elliptic then distributions, Heat eqn, Wave
Fundamental soln of wave eqn requires ↑ Then Sobolev spaces

Recall: Proved Poincare (Dirichlet version)
Remark: We shall show that on bounded domains,
$-\Delta$ has discrete spectrum.

$$0 < \lambda_1 < \lambda_2 < \dots \nearrow \infty$$

$$-\Delta\varphi = \lambda\varphi$$

φ are smooth and form a complete orthonormal basis of $L^2(\Omega)$

In a certain sense, this is a generalization of Fourier analysis to bounded domains.
It will turn out that the best value of $\delta(\Omega) = \lambda_1(\Omega)$
We have seen a geometrical description for $\delta(\Omega)$
Huge amt of research for geometrical " " of $\lambda_1(\Omega)$
One ~~can~~ cannot expect "discrete spectrum" of the Laplacian on non-compact domain.

Famous non-compact domain: $\mathbb{R}^n$

$$\varphi \in L^2(\mathbb{R}^n)$$

$$-\Delta\varphi = \lambda\varphi \quad \downarrow FT$$

$$|\xi|^2\hat{\varphi} = \lambda\hat{\varphi} \quad \rightarrow (|\xi|^2 - \lambda)\hat{\varphi} = 0$$

$\downarrow$ depends on ξ $\quad$ $\neq$ independent

$|\xi|^2 \neq \cancel{\sqrt{\lambda}}$ $\quad \hat{\varphi} = 0$

or $|\xi| = \sqrt{\lambda}$ $\quad \hat{\varphi} \neq 0$

$$\operatorname{supp}\hat{\varphi} \subseteq S^{s-1}_{\sqrt{\lambda}}$$

later show: will not be possible

Poincaré Inequality (Neumann version)

Suppose $u \in C^1(\Omega) \cap C(\bar{\Omega})$ and Ω is bounded.

Let $\bar{u} = \frac{1}{|\Omega|} \int_\Omega u$

Then

$$\|u - \bar{u}\|_{L^2(\Omega)} \leq \varepsilon(\Omega) \|\nabla u\|_{L^2(\Omega)}$$

$\hookrightarrow$ const. depending only on Ω.

Jeff Chigger: can do calculus on metric spaces satisfying volume doubling ($\frac{V_{2R}}{V_R} < R^n$) and Poincaré Ineq.

Called: (2, 2) Poincaré.

__Proof__. Cannot do right now. Need Sobolev embedding.

HW: 2: Prove for cube domain $\Omega = [-a, a]^n$

__Thm__: ~~Cacci~~ Caccioppoli ~~Theorem~~ Inequality):

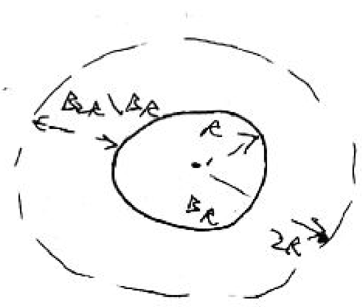
Suppose $u : B_{2R} \to \mathbb{R}$ satisfies

$u \Delta u \geq 0$ (True for harmonic functions in particular)

Then $$\int_{B_R} |\nabla u|^2 \leq \frac{4}{R^2} \int_{B_{2R} \setminus B_R} u^2$$

(~Reverse of Poincaré)

__V. Imp.__



__Example__ (Elliptic PDE "reverses" Inequality)

Holder $\|u\|_{L^2_{B(R)}} \leq C(\Omega_{B_R}) \|u\|_{L^\infty(B_R)}$

when ~~it's~~ u satisfies a "reasonable" elliptic PDE.

$$\|u\|_{L^\infty(B_R)} \leq C(B_R) \|u\|_{L^2(B_{R+\varepsilon})}$$

(may prove later)

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Elliptic : $u \in C^4 \to \Delta u \in C^2$

$\Delta u \in C^{2,\alpha} \to u \in C^{4,\alpha}$

Highest cited Book in ~~math~~

Gilbarg–Trudinger → Elliptic PDE of second order.

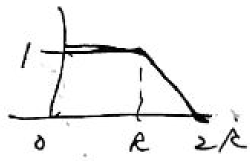
Proof: Caccioppoli Ineq.

We prove a slightly more general version:

Choose : $\varphi : B_{2R} \to \mathbb{R}$, $\varphi \geq 0$ is smooth

and $\varphi|_{\partial B_{2R}} = 0$

We will show: $\int_{B_{2R}} \varphi^2 |\nabla u|^2 \leq 4$

(taking φ :  $\int_{B_R} |\nabla u|^2 \leq \int_{B_R} |\nabla u|^2 + \int \cancel{\varphi^2} u |\nabla \varphi|^2$)

Gauss–Green Id:

$$\int_{B_{2R}} \varphi^2 u \Delta u + \int_{B_{2R}} \nabla(\varphi^2 u) \cdot \nabla u = 0$$

$$\therefore - \int \nabla(\varphi^2 u) \nabla u \geq 0 \qquad (u \Delta u \geq 0)$$

$$-2 \int_{B_{2R}} \varphi u \nabla\varphi \nabla u - \int_{B_{2R}} \varphi^2 |\nabla u|^2 \geq 0$$

$$\Rightarrow \int_{B_{2R}} \varphi^2 |\nabla u|^2 \leq -2 \int_{B_{2R}} \varphi u \nabla \varphi \nabla u$$

$$\underset{\substack{\text{~~Holder~~} \\ \text{Cauchy} \\ \text{Schwartz}}}{\leq} 2 \, |\varphi u| |\nabla \varphi| \, |\nabla u|$$

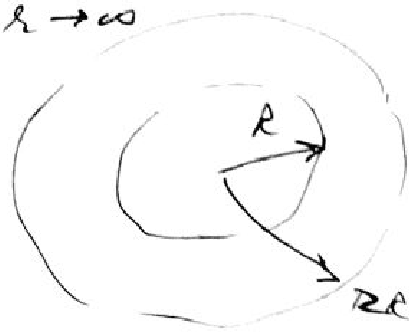
$$\underset{\text{Holder}}{\leq} \left(\int_{B_{2R}} u^2 |\nabla \varphi|^2 \right)^{1/2} \left(\int_{B_{2R}} \varphi^2 |\nabla u|^2 \right)^{1/2}$$

Divide and square.

Application: Harmonic functions must grow at some rate.

Liouville $\| \|_\infty \to \infty$ as $R \to \infty$

Now
$$\frac{\|u\|_{L^2(B_{2R})}}{\|u\|_{L^2(B_R)}}$$



L^2 can be beaten: by making one peak very narrow:

$\to L^2$ still, integrates to ~0.
L^∞ very high.

A quantitative "L^2 version" of Liouville.

Corollary $\Delta u = 0$ on $\mathbb{R}^n$

$$\int_{B_{2R}} u^2 \geq (1 + k(n)) \int_{B_R} u^2 \quad , k(n) > 0$$

Think:
$$(1 + k(n)) \int_{B_R} u^2 < \ ? \ < \int_{B_{2R}} u^2$$

$$\text{"}C\text{"} \int_{B_R} u^2 \underset{\text{Poincaré}}{\leq} \int_{B_R} |\nabla u|^2 \underset{\text{Caccio}}{\leq} \int u^2$$

Proof:
$$\int_{B_{2R}} |\nabla(\varphi u)|^2 \leq \int_{B_{2R}} |\varphi \nabla u|^2 + \int_{B_{2R}} |u \nabla \varphi|^2$$

$(\varphi \nabla u + u \nabla \varphi)^2 \to a^2 + b^2 + 2ab$

$$+ \underset{\text{Holder}}{2} \left(\int_{B_{2R}} |\varphi \nabla u|^2 \right)^{1/2} \left(\int_{B_{2R}} |u \nabla \varphi| \right.$$

Cauchy Sch. $\leq$ $2ab \leq \frac{a^2}{2} + \frac{b^2}{2}$

$$2 \int_{B_{2R}} \varphi^2 |\nabla u|^2 + 2 \int_{B_{2R}} u^2 |\nabla \varphi|^2$$

caccio gen.

$$\leq 2(4)\left(\int u^2 |\nabla \varphi|^2\right) + 2 \int u^2 |\nabla \varphi|$$

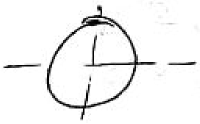
$$\leq 10 \int_{B_{2R}} u^2 |\nabla \varphi|^2$$

Poincaré:

$$\frac{1}{R^2} \int_{B_R} \varphi^2 u^2 \leq \frac{1}{R^2} \int_{B_{2R}} \varphi^2 u^2$$

$\downarrow$ Poincaré

$$\leq C \int_{B_{2R}} |\nabla(\varphi u)|^2$$



ball$_{2R}$: $\delta(\,) = (4R)^2 = 16R^2$ $\qquad C = \frac{16R^2}{R^2}$

Now take φ as in ~~the~~ earlier proof.

<u>Corollary</u>: $\Delta u = 0$ on B_{2R}

Then $\int_{B_{2R \setminus R}} |\nabla u|^2 \geq C(n) \int_{B_R} |\nabla u|^2$

sugg: ∇ Wide also harmonic.

otherwise use ~~opposite~~ "other" Poincaré.

conjecture: $\frac{\int_{B_{2R}} u^2}{\int_{B_R} u^2} \leq C R^d$ → form finite dimensional vector space.

Schrödinger operator: $-\Delta u + V u = 0$

Landis conject: cannot decay faster than $e^{-C|x|^{1+\varepsilon}}$ for real valued funcs.

$e^{-CR^{1+\varepsilon}}$

Feb 6 7pm | Midsem 25th 6:30–8:30 pm + Apt 1 week after midsem

Fourier Transform: $\hat{f}(\xi) = F(f)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-ix\xi} d\xi$

As mentioned before, $F : L^1 \to L^\infty$ bounded

Def: Schwartz space: $S(\mathbb{R}^n) : f \in C^\infty(\mathbb{R}^n) \mid x^\beta \Delta^\alpha f \in L^\infty(\mathbb{R}^n)$ $\forall \alpha, \beta \geq 0$

Not only derivative bounded but also decays faster than $\frac{1}{x^\beta}$

Ex: $x_3^{10} \frac{\partial^{15}}{\partial x_2^{10} \partial x_3^{5}} f(x_1, x_2, x_3) \in L^\infty(\mathbb{R}^n)$ $\alpha = (0, 10, 5)$, $\beta = (0, 0, 10)$

Ex $C_c^\infty(\mathbb{R}^n) \subseteq S(\mathbb{R}^n)$ → supp, big $x^\beta(\cdot\,0)$ can be bounded

$e^{-|x|^2} \in S(\mathbb{R}^n)$

Inverse Fourier Transforms

$F^* f(\xi) = F^{-1} f(\xi) = \frac{-1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{ix\cdot\xi} dx$

Homework:

a) $F(S(\mathbb{R}^n)) \subseteq S(\mathbb{R}^n)$

$F^*(S(\mathbb{R}^n)) \subseteq S(\mathbb{R}^n)$

~~Hint~~ $\xi^\alpha \Delta_\xi^\beta F(f)(\xi) = (-1)^{|\beta|} (-1)^{|\alpha|} F(\Delta_x^\alpha x^\beta f)(\xi)$

multiplication by polyn $\overset{\text{exchange}}{\sim}$ differentiation $\xi^\alpha \Delta^\beta \to \Delta^\alpha x^\beta$

$\Delta_x^\alpha (x^\beta f) = \sum_\delta \sum_\gamma x^\gamma \Delta^\delta f$ all fall in Schwartz

Thm F, F^* are 1-1 on $S(\mathbb{R}^n)$

later (isometry) Also S is dense L^2

Heat Equation: $\frac{\partial u}{\partial t} = \Delta u$ on $\mathbb{R}^+ \times \mathbb{R}^n$

$u(0, x) = f(x) \to$ Initial condition

Infinite speed of propagn: ~~Start~~ with initial condn that is compactly supported, for any $\varepsilon > 0$ (t) the soln is non-zero almost everywhere

Assume: f, $\hat{u}$ make sense

Remark For now this can mean that $u, f \in S(\mathbb{R}^n)$. but we will later on extend this to include at least tempered distributions.

$$\partial_t \hat{u} = -|\xi|^2 \hat{u}$$ (Taking Fourier Transform with respect to x-variable

$$\hat{u}(0,\xi) = \hat{f}(\xi)$$

solving 1st order ODE: $\hat{u}(t,\xi) = e^{-t|\xi|^2}\hat{f}(\xi)$

$$u(t,x) = \frac{1}{(2\pi)^{n/2}} F^{-1}(e^{-t|\xi|^2}) * f(x)$$ (inverse FT of both convolve

Facts: ~~$F(f$~~ ~~$F(fg)$~~ $\widehat{fg} = c\hat{f}(\xi) * \hat{g}(\xi)$

conver $F^* = F^{-1}$

$\boxed{F(u*v) = F(u)F(v)(2\pi)^{n/2}}$ $\widehat{f*g}(\xi) = c\hat{f}(x)\hat{g}(x)$

<u>check</u> $\frac{1}{(2\pi)^{n/2}} F^{-1}(e^{-t|\xi|^2}) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}$

FT(gaussian) $\rightarrow$ gaussian.

$$u(t,x) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t} * f(x)$$

$$= \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} e^{-|x-y|^2/4t} f(y)\,dy$$

Heat kernel: $\frac{1}{(4\pi t)^{n/2}} e^{-|x-y|^2/4t} = p(t,x,y)$

Derive gauss! $F(\xi) = \int_{\mathbb{R}} e^{-x^2} e^{-ix\xi}\,dx$

$$F'(\xi) = \int_{\mathbb{R}} e^{-x^2}(-ix)\,e^{-ix\xi}\,dx$$

$$= \frac{i}{2}\int (e^{-x^2} - 2x)$$

$$= \frac{i}{2}\int_{\mathbb{R}} \frac{d(e^{-x^2})}{dx} e^{-ix\xi}\,dx$$

$$= \frac{i}{2}\int e^{-x^2}(-i\xi)\,e^{-ix\xi}\,dx$$

$$\boxed{F'(\xi) = -\frac{\xi}{2} F(\xi)}$$ $\rightarrow$ solve ODE

$$F(0) = \int_{\mathbb{R}} e^{-x^2}\,dx = \sqrt{\pi}$$

Properties of Heat Kernel

Prof Gopal Show, Expert on Gamma func

$p(t, x, y) \geq 0 \quad \forall\, t > 0,\, x, y$

· "decays very fast spatially

1. $\int_{R^n} p(t, x, y)\, dy = 1$ standard Normal distribution

$\frac{1}{(4\pi t)^{n/2}} \int_{R^n} e^{-|x-y|^2/4t}\, dy$

x is fixed

"origin" $e^{-|y|^2/4t}$ radial about x.

$y \to y_1, y_2, \ldots, y_n$

$dy_1 dy_2\, dy \ldots dy_n$

$\frac{y-x}{2\sqrt{t}} = r\omega$

$dy = 2\sqrt{t}\, d(r\omega)$

$dy = (4t)^{n/2} r^{n-1} dr\, d\omega$

$= \frac{1}{\pi^{n/2}} \int_0^\infty \int_{S^{n-1}} e^{-r^2} r^{n-1}\, dr\, d\omega$

$= \frac{1}{\pi^{n/2}} \int_{R^n} e^{-|z|^2}\, dz = 1$

+ product of n integrals $(\sqrt{\pi})^n$

2. $\int_{R^n} p(t, x, y) f(y)\, dy$

What happens when $t \to 0$

if $f: R^n \to R$ is bounded and continuous

then $\lim_{t \to 0} u(t, x) = f(x)$

Proof $|f(x) - u(t, x)|$ "integrates to 1"

Same as that for elliptic Neumann kernel

$= \left| \int_{R^n} f(x)\, p(t, x, y)\, dy - \int_{R^n} f(y)\, p(t, x, y)\, dy \right|$

$= \left| \int_{R^n} (f(x) - f(y))\, p(t, x, y)\, dy \right|$

Quiz Sat 16/2/2019
Fri Problem sess 15/2/2019

$$\frac{x-y}{\sqrt{t}} = rw$$

$$\leq \frac{1}{(4\pi)^{n/2}} \left| \int_{S^{n-1}}\int^{\infty} (f(x) - f(x-\sqrt{t}rw)) e^{-4r^2} r^{n-1} dr dw \right|$$

$$\leq \frac{1}{(4\pi)^{n/2}} \int_{S^{n-1}}\int_0^{\infty} |f(x) - f(x-\sqrt{t}rw)| e^{-4r^2} r^{n-1} dr dw$$

$$\leq \frac{1}{(4\pi)^{n/2}} \int_0^{K}\int_{S^{n-1}} |f(x) - f(x-\sqrt{t}rw)| e^{-r^2} r^{n-1} dr dw$$

$$+ \frac{1}{(4\pi)^{n/2}} \int_{K}^{\infty}\int_{S^{n-1}} |f(x) - f(x-\sqrt{t}rw)| e^{-r^2} r^{n-1} dr dw$$

if f bounded $\leq 2M$

second term

$$II) \underset{(const)}{\lesssim} 2M \int_K^{\infty}\int_{S^{n-1}} e^{-r^2} r^{n-1} dr dw$$

$$\sim A_{S^{n-1}} \int_K^{\infty} e^{-r^2} r^{n-1} dr \quad \text{Take } r \text{ large enough}$$

$$e^{-r^2} r^{n-1} \leq e^{-r^2/2}$$

$$\lesssim \int_{K_2}^{\infty} e^{-r^2} dr \rightarrow \text{tail of gamma func}^n. \text{ integral}$$

Even if entire $\rightarrow$ bounded $\sqrt{\pi}$

Choose K large enough such that $II < \varepsilon/3$

then choose t small enough s.t. $I < \frac{\varepsilon}{3}$

comments $\int_{\mathbb{R}^n} p(t, x-y) f(y) dy$ is smooth

$t \quad e^{-|x|^2/4t} * f(x)$ smooth

$x \in \mathbb{R}^n$

$z \in \mathbb{C}^n$

$t > 0 \quad \mathrm{Re}\, z > 0$

Real analytic

(can't vanish on an open set)

By Taylor

$\therefore$ Has to be non-zero everywhere $\rightarrow$ inf. speed of propaga

Backtracking not possible

Feb 8 Fri 3:30pm **Recall**. We have constructed solutions to the heat equations

$$\left.\begin{aligned}\partial_t u - \Delta u &= 0\\ u(0,x) &= f(x)\end{aligned}\right\}$$

as
$$u(t,x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-|x-y|^2/4t} f(y)\,dy$$

define
$$e^{t\Delta} f(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-|x-y|^2/4t} f(y)\,dy$$

Linear
$$e^{t\Delta}(f(x)+g(x)) = \frac{1}{(4\pi t)^{n/2}} \int e^{-|x-y|^2/4t} (f(y)+g(y))\,dy$$

$$= \frac{1}{(4\pi t)^{n/2}} \int f(y) e^{-|x-y|^2/4t} dy + \int g(y) e^{-|x-y|^2/4t} dy$$

$$e^{t\Delta}(f(x)+g(x)) = e^{t\Delta} f(x) + e^{t\Delta} g(x)$$

__Will prove__: __Def__ $\|Tf\|_{L^2} \le \|f\|_{L^2}$ (contractive)

$e^{t\Delta}$ is a contractive semigroup on L^2

$e^{t\Delta}$ is also called "heat semigroup"

__semigroup property__

$$e^{(t_1+t_2)\Delta} = e^{t_1\Delta} e^{t_2\Delta} \qquad t_1, t_2 \ge 0$$

We can define functions of symm matrices: (symm → diagonaliz)

similarly: functions of self adjoint lin operator

spectral theorem for unbounded lin operators

$\mathrm{sqrt}(\Delta)$, $e^{it\sqrt{\Delta}}$, ...

$h(x) = e^{tx}$

$h(\Delta) \to e^{t\Delta}$

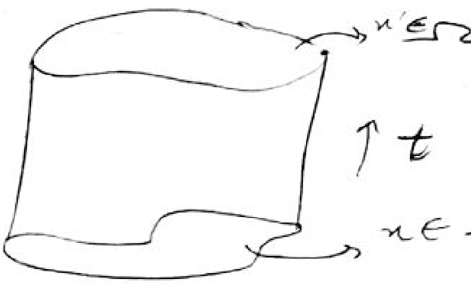
PDEs: unbounded operator.

Natural Hilbert Space L^2

Parabolic Maximum Principle on bounded domain

Temp distrib^n. High to low.

let $\Omega_T = (0, T) \times \Omega$



$\Gamma_T = ([0,T] \times \partial\Omega) \cup (\{0\} \times \overline{\Omega})$ → bottom

→ $t=0$ gives I.C. wrt t.

"Reduced Boundary"

Assume $u(\cdot, x) \in C^2_x(\Omega)$

$u(t, \cdot) \in C^1_T((0,T))$

$\partial_t u = \Delta u$

$u(t,x) = f(t,x)$ on Γ_T (Reduced Boundary cond^n)

(Eigenvalues of Δ on $\bigcirc$ $u(t,x) = \sum_j \varphi_j(x) e^{-\lambda_j t}$)

Conclusion

(1) $\partial_t u - \Delta u \le 0$ in Ω_T, then

$$\max_{\overline{\Omega}_T} u = \max_{\Gamma_T} u$$

(max overall is on both bd. and int.)

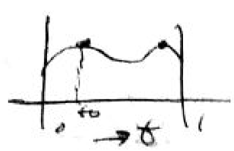
(2) $\partial_t u - \Delta u \ge 0$ in Ω_T then,

$$\min_{\overline{\Omega}} u = \min_{\Gamma_T} u$$

Proof (1) suppose $\partial_t u - \Delta u < 0$ ① Rabble

let $\exists\ (t_0, x_0) \in \overline{\Omega}_T \setminus \Gamma_T$ (interior) such ~~that~~

$$u(t_0, x_0) = \max_{\overline{\Omega}_T} u$$



~~then~~ If $t_0 < T$ at (t_0, x_0),

$\Delta u < 0$ (max) $\quad \partial_t u\big|_{t=t_0} = 0$ (max)] contradic^n ①

If $t_0 = T$ $\Delta u \le 0$, $\partial_t u\big|_{T = t_0} \ge 0$

$\partial_t u - \Delta u \leq 0$

$u_\varepsilon := u - \varepsilon t \quad , \varepsilon > 0$

$\partial_t u_\varepsilon - \Delta u_\varepsilon = -\varepsilon \quad < 0$

$\max_{\bar{\Omega}_T} u_\varepsilon = \max_{\Gamma_T} u_\varepsilon$ max over entire domain has to be attained at Boundary

$\Downarrow \quad \varepsilon \to 0$ pointwise convergence

$\max_{\bar{\Omega}_T} u = \max_{\Gamma_T} u$

For min: $u_\varepsilon := u + \varepsilon t$

Remark Parabolic maximum principle would have worked for $\partial_t u - Lu = 0$, where L is some Diff operator for which $Lu \leq 0$ at local max pts.

Eg: $L = \underset{+>0}{e^{x_1}} \partial^2_{x_1} + \underset{>0}{\sin^2 x_2}\, \partial_{x_2}$ if $\partial^2_{x_1}, \partial^2_{x_2} < 0$ then also < 0

(Only property of Laplacian used in prev. proof is $\Delta u < 0$ for max)

Remark: Parabolic maximum Principle establishes uniqueness of solutions of heat equation in $C^1_t \times C^2_x$

(subtract $\to$ 0 at boundary: max at boundary $\to \max \tilde{\varphi} = 0$
min " " $\min \tilde{\varphi} = 0$
$\to \tilde{\varphi} = 0$ everywhere
$\varphi_1 - \varphi_2 = 0$
$\varphi_1 = \varphi_2 \to$ unique

Proposition

$\partial_t u = \Delta u, \quad u \in \mathcal{S}(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n)$

$u(0, x) = 0$

$\mathcal{S} \subseteq L^2$
$\int u^2 r^{n-1} dr\, d\omega \to \frac{x^{high} u^2}{x^{high}} r^{n-1}$ $x^{high} > n$

Then $u = 0$

$\partial_t \int_{\mathbb{R}^n} u^2(t,x)\, dx = \int_{\mathbb{R}^n} \partial_t u^2(t,x)\, dx = \int_{\mathbb{R}^n} 2u\, \partial_t(u)$

$= \int_{\mathbb{R}^n} 2u \Delta u$

$\partial_t \int_{\mathbb{R}^n} u^2 dx = -2 \int |\nabla u|^2 \leq 0$ (Gauss Green Identity) 0 at Boundary

L^2 norm is ~~decreasing~~ non increasing ~~with~~ time.

contractive $\| e^{t\Delta} f \| \leq \|f\|$

if we start with norm 0 $\to$ stays 0

Now we go for much stronger uniqueness results.

Thm Uniqueness on $\mathbb{R}^n$ for sufficiently fast growing solutions.

~~$u \in C^2_x(\mathbb{R}^n) \wedge C_t$~~ $u \in C^1_t \times C^2_x$

$\partial_t u - \Delta u \le 0$ on $(0,T) \times \mathbb{R}^n$

$u(t,x) \le A e^{a|x|^2}$ on $(0,T) \times \mathbb{R}^n$

$a, A > 0$

This drastically improves our previous assumption that $u \in S(\mathbb{R}^n)$

Then $\sup_{(0,T) \times \mathbb{R}^n} |u(t,x)| \le \sup_{\mathbb{R}^n} |u(0,x)|$

Remark: There are counterexamples of the following kind

$\partial_t u = \Delta u$

$u(0,x) = 0$

$u(t,x) \ne 0$

$u(t,x) >> \alpha e^{\beta |x|^2}$ $\beta > 0$

Tychonoff 1930s

Fritz John's book

Feb 13 Wed 7pm • Recall: Soln to ~~heat~~ Eqn is unique under

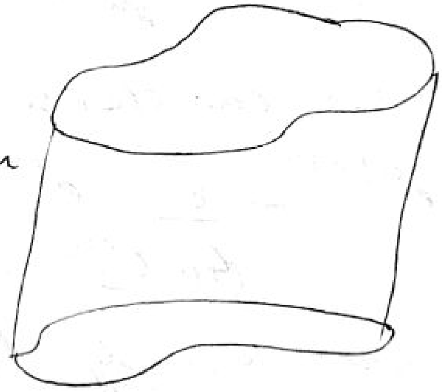
$\partial_t u - \Delta u = 0$ $u(t,x) \le \cancel{A}e^{|x|^2}$

$$\sup_{(0,T) \times \mathbb{R}^n} |u(t,x)| = \sup_{\mathbb{R}^n} |u(0,x)|$$

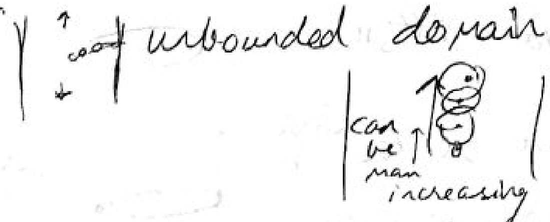
Parabolic Principle

look up Auxilliary func which decays away from origin

The $u - A\rho$'s also solves Heat Eqn.



HW2 Thm ~~cool~~ unbounded domain problem



• Proof Check that for $\tau > 0$

$$\rho(t,x) = \frac{1}{[4\pi(T-t)]^{\frac{n}{2}}} e^{-\frac{|x|^2}{4(\tau-t)}}$$

also solves ~~the~~ heat Eqn

Let $v(t,x) = u(t,x) - \varepsilon \rho(t,x)$ $0 \le t \le \frac{T}{2}$

Let $v(t,x) = u(t,x) - \varepsilon \rho(t,x) \qquad 0 \le t \le \frac{\tau}{2}$

$$\partial_t v(t,x) = \Delta v(t,x)$$

$$v(t,x) \le A e^{a|x|^2} - \frac{\varepsilon}{[4\pi(\tau-t)]^{n/2}} e^{|x|^2/4(\tau-t)}$$

$$\le A e^{a|x|^2} - \frac{\varepsilon}{(4\pi\tau)^{n/2}} e^{|x|^2/4\tau}$$

$$\left(\because \frac{e^{|x|^2/4\tau}}{(4\pi\tau)^{n/2}} \le \frac{e^{|x|^2/4(\tau-t)}}{(4\pi(\tau-t))^{n/2}}\right)$$

div by small

divide by smaller no

choose $\tau < \frac{1}{4a}$

$a < \frac{1}{4\tau}$

for ~~a set~~ $\frac{1}{4\tau} > a \qquad e^{a|x|^2} < \frac{\varepsilon e^{|x|^2/4\tau}}{(4\pi\tau)^{n/2}}$

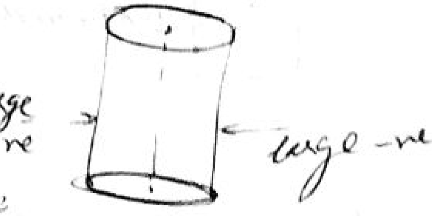
suppose $\sup_{\mathbb{R}^n} \{u(0,x)\} = k \ge 0$

Then we can choose $M > 0$ sufficiently large s.t.

$A e^{a|x|^2} - \frac{\varepsilon}{(4\pi\tau)^{n/2}} e^{|x|^2/4\tau}$ is large

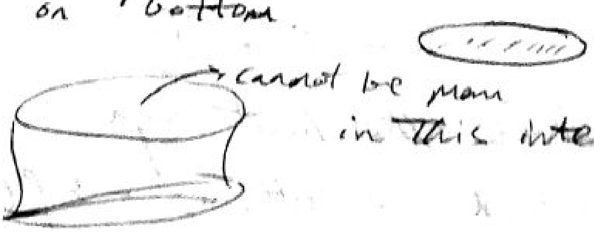
We want to apply Parabolic max principle on

$(0, \frac{T}{2}) \times B_M(0)$ for $v(t,x)$



∴ max can't be on sides of cylinder ∵ large -ve value

∴ Only region where it can attain max is on bottom



cannot be max in this interior (Max principle on bounded domain)

$$\sup_{(0,T/2)\times B_R(0)} v(t,x) = \sup_{\mathbb{R}^n} v(0,x)$$

Schrö do same.
$FT \to ODE$
$+ \text{inv} FT$

$\varepsilon \to 0$, $\le \sup_{\mathbb{R}^n} u(0,x) = K$

$$\sup_{(0,T/2)\times B_R(0)} u(t,x) \le K$$

Do this inductively $[T/2, T]$, $[T, 3\frac{T}{2}]$ ---
till T is reached.

- Kill solⁿ to make compact ~~suppor~~.
- $\varepsilon \to 0$

- **Distributions**

Recall: Schwartz space.

F^* is adjoint

Thm $F^*F = FF^* = I$ on $S(\mathbb{R}^n)$

(Plancheral follows by dense of S in L^2)
$F(u) \in S(\mathbb{R}^n)$
$F(u) \in L^2$

Plancheral: $F: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is an isometry
$(Fu, v) = (u, Fv)$
In Hilbert space norm $\to$ inner prod

Proof
$$F^*F(f(x)) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-iy\cdot\xi} f(y)\,dy\; e^{ix\cdot\xi}\,d\xi$$

$$= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} f(y)\,dy\,d\xi$$

$$= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} \lim_{\varepsilon\to 0} e^{-\varepsilon|\xi|^2} f(y)\,dy\,d\xi$$

Justify DCT
$$= \frac{1}{(2\pi)^n}\lim_{\varepsilon\to 0}\iint e^{i(x-y)\cdot\xi-\varepsilon|\xi|^2} f(y)\,dy\,d\xi$$

We want
LHS $= f(x)$
We know
$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} p(\varepsilon,x,y) f(y)\,dy = f(x)$$
Heat kernel

Define $p(\varepsilon, x) = \frac{1}{(2\pi)^{n/2}} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-\varepsilon|\xi|^2} e^{ix\cdot\xi} d\xi$

$\leftarrow$ is FT of Gauss.

$$= \frac{1}{(4\pi\varepsilon)^{n/2}} e^{-|x|^2/4\varepsilon}$$

$$p(\varepsilon, x-y) = \frac{1}{(4\pi\varepsilon)^{n/2}} e^{-|x-y|^2/4\varepsilon}$$

$$= \int_{\mathbb{R}^n} e^{-\varepsilon|\xi|^2} e^{i(x-y)\cdot\xi} d\xi$$

<u>Pseudo diff operators</u>

$$\Delta u = f$$

FT

$$-|\xi|^2 \hat{u} = \hat{f}$$

$$\hat{u} = -\frac{1}{|\xi|^2} \hat{f}$$

$\cancel{I}$FT

$$u = -\widehat{\frac{1}{|\xi|^2}} * f$$

$$\int_{\mathbb{R}^n} \frac{1}{|\xi|^2} e^{ix\cdot\xi} \hat{f}(\xi) d\xi$$

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{1}{|\xi|^2} e^{i(x-y)\cdot\xi} f(y)\, dy\, d\xi$$

general Pseudo diff operator:

$$p(x, D) f = \iint p(x,\xi) e^{i(x-y)\xi} f(y)\, dy\, d\xi$$

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Feb 15 Fri 3:34pm Corollary of Harnack Liouville's thm. Suppose $u \in C^2(\mathbb{R}^n)$ is harmonic on $\mathbb{R}^n$. Assume $\exists$ a constant M such that $u(x) \geq M$ $\forall x \in \mathbb{R}^n$ or such that $u(x) \leq M$ $\forall x \in \mathbb{R}^n$. Then u is a constant function.

Proof. $u(x) \geq M$. Let $v = u + |M|$

$v \geq 0$ is harmonic ~~and~~

$x \in \mathbb{R}^n$, R is sufficiently large

$$\frac{R^{n-2}(R-|x|)}{(R+|x|)^{n-1}} v(0) \leq v(x) \leq R^{n-2}\frac{(R+|x|)v(0)}{(R-|x|)^{n-1}}$$

$R \to \infty$

$v(x) = v(0)$

for $u(x) \leq M$ consider $v = -u(x) + |M|$

HW 2

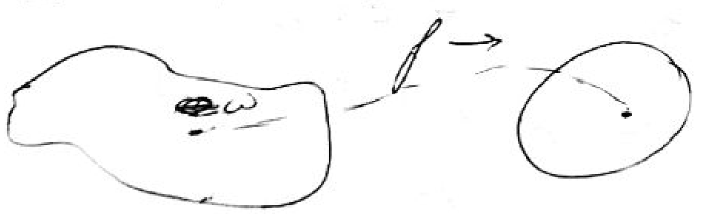
1) Riemann mapping thm. Any simply connected domain which is not $\mathbb{C}$ is bianalytically equivalent to $B_1(0)$.

one to one holomorphic with inverse.

$f: B_1(0) \to \mathbb{C}$ $\quad$ $f^{-1}(\mathbb{C}) \to B_1(0)$ bounded

bianalytic = biholomorphic $\quad$ By Liouville. $f^{-1} = c$

$f\ \Omega \to B_1(0)$



$f(w) = 0$

$f(\partial\Omega) \xrightarrow{?} \partial B_1(0)$ $\quad$ $f(z) = (z-w)e^{g(z)}$

$|f(z_{\partial B})| = 1$ $\quad$ $g(z) = u + iv$

$|f(z_{\partial B})| = |(z-w)| e^u = 1$

Dirichlet problem $\left| \begin{array}{l} \Delta u = 0 \\ u|_{\partial\Omega} = -\ln(|z-w|) \end{array} \right.$ $\quad$ $e^u = \frac{1}{|z-w|}$

$u = \ln \frac{1}{|z-w|}$

since determine

on a simply connected domain,

$f \neq 0 \Rightarrow f = e^{g(z)}$

2) $u = xy(\ln x)^\beta$

$\partial_x u = y(\ln x)^\beta + \beta(\ln x)^{\beta-1}$

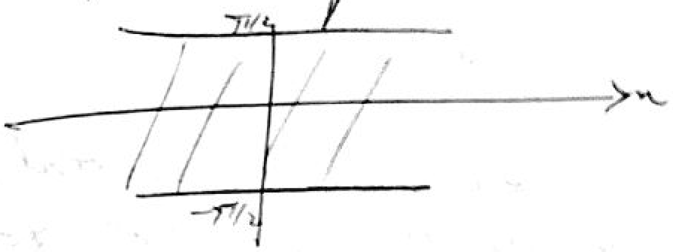
3) $\Delta v = 0$ on B_{2R}

$\sup_{B_R(0)} v \leq 3^{3n} \inf v$

$R \to \infty$

Hopf Boundary Lemma

1)



Rudin Complex and Real

f is holomorphic on S

" cont on $\bar{S}$

for all $z \in S$, $|f(z)| < e^{Ae^{c|\operatorname{Re} z|}}$, $A < \infty$, $c < 1$

and $|f(z)| \leq 1 \quad \forall z \in \partial S$

claim $|f(z)| \leq 1 \quad \forall z \in S$

proof $h_\varepsilon(z) = e^{-\varepsilon(e^{\beta z} + e^{-\beta z})}$ $\quad \varepsilon > 0$

where β is selected s.t.

$0 < c < \beta < 1$

$z = x + iy$ $\quad e^{\beta z} + e^{-\beta z} = e^{\beta x} e^{iy} + e^{-\beta x} e^{-i\beta y}$

$= \underbrace{(e^{\beta x} + e^{-\beta x})\cos\beta x}_{X} + \underbrace{i(e^{\beta x} - e^{-\beta x})\sin\beta y}_{Y}$

$|h_\varepsilon(z)| = |e^{-\varepsilon(X+iY)}|$

$= e^{-\varepsilon X}$

$= e^{-\varepsilon[(e^{\beta x} + e^{\beta x})\cos\beta y]}$

$\underline{\beta < 1 \quad \beta y < \frac{\beta\pi}{2}}$

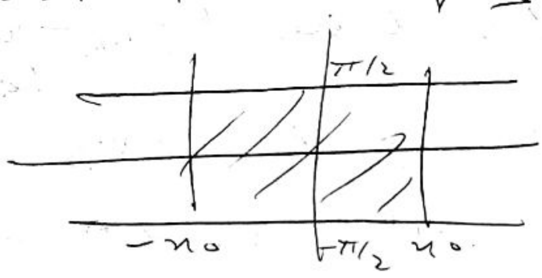
$\cos\beta y \ge \cos\beta\frac{\pi}{2}$

$$\boxed{|h_\varepsilon(z)| \le e^{-\varepsilon(e^{\beta x}+e^{-\beta x})\cos(\beta\frac{\pi}{2})}}$$

$$|f h_\varepsilon(z)| \le e^{Ae^{cx} - \varepsilon\cos\beta\frac{\pi}{2}(e^{\beta x}+e^{-\beta x})}$$

fix $\varepsilon > 0$ we can get x_0 st. $\cancel{AA}e^{cx_0} - \varepsilon\cos\beta\frac{\pi}{2}(e^{\beta x_0}+e^{-\beta x_0}) \le 0$

$|f h_\varepsilon(z)| \le 1 \quad \forall\, S$



Apply Max mod now

Feb Quiz Q8
20 wed
$$u(t,x) = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t} f(y)\,dy$$

Fritz John's Book Expansion.

$x \mapsto z \quad \operatorname{Re}(z) = x$

$$u(t,z) = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-(z-y)^2/4t} f(y)\,dy$$

$$0 = \int_\Gamma u(t,z)\,dz = \int_{\mathbb{R}}\int_\Gamma \frac{1}{\sqrt{4\pi t}} e^{-(z-y)^2/4t} f(y)\,dy$$

$\square$

MODERN THEORY OF PDE (MA 534) MID-SEMESTER EXAM

(1) Calculate the Fourier transform of $\chi_{[-1,1]}(x)$, the characteristic function of the closed interval $[-1,1] \subset \mathbb{R}^n$. How does your answer decay at infinity? Is it in the Schwartz class? 3 + 3

(2) Write the Laplacian in polar coordinates in $\mathbb{R}^2$ (you do not need to show computations, you can quote the formula from memory). Find all radial harmonic functions on $\mathbb{R}^2 \setminus B_1(0)$ which have the boundary value 1 on $\partial B_1(0)$. 3 + 3

(3) (a) Establish the following *maximum principle* for subharmonic functions for bounded domains $\Omega \subset \mathbb{R}^n$: let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy $\Delta u \geq 0$. Then we have

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u.$$

5

(b) Can you give an example where the maximum principle fails for harmonic functions on domains Ω which are not the whole of $\mathbb{R}^n$? (Note: You don't need to prove your counterexample works, just state it. If you need a hint, think Phragmen-Lindelöf type results). 2

(4) Let $u(t,x) = \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{(x-y)^2}{4t}} f(y)dy$ solve the heat equation $\partial_t u = \Delta u$ on $\mathbb{R}^n$ with initial condition $u(0,x) = f(x)$. If $f \in C_c^\infty(\mathbb{R}^n)$, then prove that $u(t,x)$ extends to a complex analytic function of $x \in \mathbb{C}^n$ and $t \in \mathbb{C}$ such that $\mathrm{Re}\, t > 0$. 6

$$\int_{-1}^{1} \cos(2\pi x\xi)\, dx + i \int_{-1}^{1} \sin(2\pi x\xi)\, dx$$

$$+\frac{\sin 2\pi x\xi \Big|_{-1}^{1}}{(2\pi\xi)} + \frac{i(-\cos 2\pi x\xi)\Big|_{-1}^{1}}{(2\pi\xi)}$$

$$= \frac{2\sin(2\pi\xi)}{2\pi\xi} + i\, \cancel{\cos(2\pi\xi)} \to 0$$

$$= \frac{\sin(2\pi\xi)}{\pi\xi}$$

~~Let~~ $\int e^{z} dz$

$e^{z}\Big|_{-1}^{1}$

$\frac{e^{z} - e^{-z}}{2}$

$2i \quad \frac{e^{2\pi i\xi} - e^{-2\pi i\xi}}{(2\pi i\xi)}$

$r = (x^2+y^2)^{1/2}$

$\frac{\partial x}{\partial r} \quad \frac{\partial x}{\partial \theta}$

$\frac{\partial y}{\partial r} \quad \frac{\partial y}{\partial \theta}$

$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

$r\cos^2\theta + r\sin^2\theta$

$\partial r = \frac{\partial r}{\partial x}\, dr \qquad d\theta$

$r = r(x,y)$

$\hat{r} = dr$

$\frac{dr}{dr} = \frac{\partial r}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial r}{\partial y}\frac{\partial y}{\partial r}$

$= \cos\theta$

$r\,dr$

$d\theta$

~~$\theta + d\theta$~~

$r + dr =$

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· Last Class

$F : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ is an isometry

Remark : Observe ~~that~~ F^* does not provide inversion for $F : L^1 \longrightarrow L^\infty$

This will follow as a biproduct of fourier inversion formula for tempered distribution.

for $L^2 \longrightarrow L^2$ vs but $L^1 \longrightarrow L^\infty$

$F(\xi) \longrightarrow \hat{f}$ inv.

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-i x \cdot \xi} dx$$

$$\underbrace{|\hat{f}(\xi)|}_{L^\infty} \le \frac{1}{(2\pi)^{n/2}} \underbrace{\int_{\mathbb{R}^n} |f(x)| dx}_{L_1}$$

Triangle inequality

Constant function c ——— in L^∞ not in L^1

Special: Tempered distributions

Work very well with fourier transforms
All cont. linear functionals $S(\mathbb{R}^n) \to \mathbb{C}$ or
denoted by $S^*(\mathbb{R}^n)$

* What is the topology on $S(\mathbb{R}^n)$? Continuity requires def. Top.

$\forall\ \varphi \in S(\mathbb{R}^n)$ define $\|\varphi\|_k = \max\limits_{|\alpha|+|\beta| \le k} \sup |x^\alpha \partial^\beta \varphi|$

Take two functions $\phi, \psi \in S(\mathbb{R}^n)$

$x^\alpha \partial^\beta \varphi \to x^\alpha \partial^\beta \psi$ should be close to each other

define one norm: $\|\varphi\|_k = \max\limits_{|\alpha|+|\beta| \le k} \sup |x^\alpha \partial^\beta \varphi|$

Metric space:

$$d(\varphi, \psi) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{\|\varphi - \psi\|_k}{1 + \|\varphi - \psi\|_k} \to < \frac{1}{2^k}$$

converges

Not a normed space: $\|\varphi\|_k$ Not Normable: cannot find unique norm which gives same topology

Frechét spaces: topological vector spaces + metric space complete which is translation invariant.

d does not satisfy norm conditions: doesnt scale.

Example $C_c^\infty(\Omega)$ Ω bdd. domain in $\mathbb{R}^2$

$u, v \in C_c^\infty(\Omega)$

close to each other if close to each other in all derivatives.

$\| u - v \|_{C^k}$ C^k is norm deriv $\partial^k(u)$

problem: may be close up to 1st million deriv. but next one may not.

$$d(u, v) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{\|u - v\|_{C^k}}{1 + \|u - v\|_{C^k}}$$

Usual norms $L^1, L^2, \ldots L^\infty$ don't work for closeness of smooth functions.

To capture smoothness func^n: make derivatives close.

- Much harder : $C_c^\infty(\mathbb{R}^n)$: What is the topology? difficult to justify the definition.

Here functions don't have the same compact support as before.

seminorm: $p_k(u) = \|u\|_{C^k}$

Embedding
$$C_0^\infty(B_R(0)) \xrightarrow{i_R} C_c^\infty(\mathbb{R}^n)$$

- The topology on $C_c^\infty(\mathbb{R}^n)$ is defined as the finest topology ~~that~~ makes each i_R continuous (inductive limit topology)

Topology is finer if it contains all open sets of previous one.

- Now ~~that~~ we have topology on $S(\mathbb{R}^n)$ give topology on $S^*(\mathbb{R}^n)$

- Weak Topology : $\omega_n, \omega \in S^*(\mathbb{R}^n)$

$$\omega_n \longrightarrow \omega \overset{\text{define}}{\Longleftrightarrow} \underbrace{\omega_n(u)}_{\substack{\text{dual pairing} \\ \text{complex No.}}} \longrightarrow \omega(u) \quad \forall u \in S(\mathbb{R}^n)$$

*complex No.

Choose $u \in S(\mathbb{R}^n)$

$\omega \in S^*(\mathbb{R}^n)$ s.t. $|\omega(u)| < \varepsilon$

Eg: $C_c^\infty(\mathbb{R}^n) \subseteq S^*(\mathbb{R}^n)$

pick $\varphi \in C_c^\infty(\mathbb{R}^n)$, $u \in S(\mathbb{R}^n)$

$$\varphi(u) := \int_{\mathbb{R}^n} \varphi u$$

$u_n \to u$ in $S(\mathbb{R}^n)$ $\qquad \varphi(u_n) - \varphi(u) = \int_{\mathbb{R}^n} \varphi(u_n - u)$

$\varphi(u_n) - \varphi(u) = \int_{\mathbb{R}^n} \varphi(u_n - u)$

$\leq \|u_n - u\|_{L^\infty} \|\varphi\|_{L^1}.$

Also works for $S(\mathbb{R}^n) \subseteq S^*(\mathbb{R}^n)$ $\quad C_c^\infty \subseteq S(\mathbb{R}^n)$

pick $\varphi \in S(\mathbb{R}^n)$, $u \in S(\mathbb{R}^n)$.

- Eg <u>Dirac Delta distribution</u>

$S^*(\mathbb{R}^n) \ni \delta_x$. $\quad \forall u \in S(\mathbb{R}^n)$

give action on some schwartz space.

$\delta_x(u) := u(x)$

Captures impulse

Complex valued to take care of fourier $e^{i\xi x}$

- <u>Continuity</u>: $u_n \to u$ $\quad \|u_n - u\|_{L^\infty} \to 0$

$u_n(x) \to u(x)$ ← equivalent

Tempered distributions: largest class of solutions which satisfy PDEs with good properties with Fourier Transf

- Ex) $L^p(\mathbb{R}^n) \supseteq S(\mathbb{R}^n)$ $\quad 1 \leq p < \infty$

$\cap$

$S^*(\mathbb{R}^n)$ Using Holder

- <u>Derivative of tempered Distribution</u>

Let $f \in S(\mathbb{R}^n)$ $\quad w \in S^*(\mathbb{R}^n)$

$\partial_j w.f := (w, -\partial_j f)$

(Integration by parts) (without surface terms) → decay at $\pm\infty$

$\partial_j w(f) := w(-\partial_j f)$

$w(f)$ + (w, f) Duality pairing

• Some comments on Fourier Transform

Interpolation theorems: $T : X_1 \to Y_1$ bounded linear operator.

X_1, X_2 Banach spaces. $T : X_2 \to Y_2$

Y_1, Y_2

• Marcinkiewicz Interpolation theorem (Not in our course → Harmonic Analysis)

• Riesz-Thorin Interpol^n thm

Suppose: $T : L^{p_0} \to L^{q_0}$ is bounded

$T : L^{p_1} \to L^{q_1}$ are bounded

$\begin{pmatrix} p_0 \le p_1 \\ q_0 \le q_1 \end{pmatrix}$ Then: $\dfrac{1}{p_\theta} = \dfrac{1-\theta}{p_0} + \dfrac{\theta}{p_1}$

(In L^p: Hölder $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$) $\dfrac{1}{q_\theta} = \dfrac{1-\theta}{q_0} + \dfrac{\theta}{q_1}$

• <u>Claim</u>: $T : L^{p_\theta} \to L^{q_\theta}$ $\quad 0 \le \theta \le 1$

is bounded with norm:

$$\|T\|_{L^{p_\theta} \to L^{q_\theta}} \le \|T\|^{1-\theta}_{L^{p_0} \to L^{q_0}} \|T\|^{\theta}_{L^{p_1} \to L^{q_1}}$$

$F : L^2 \to L^2, \quad L^1 \to L^\infty$

$F : L^p \to L^q \qquad \frac{1}{p} + \frac{1}{q} = 1$

$1 \le p \le 2$ $\qquad 2 \le q \le \infty$

⊛ $F(L^p) \not\subseteq L^q \qquad p > 2$

Ref: Hormander Vol I Ch 7 §7.6

• Interesting fact

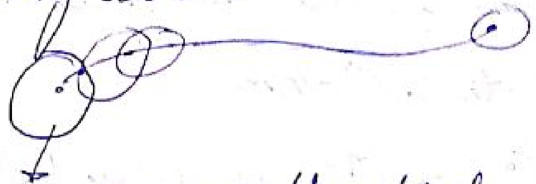
$(L^p)^* = L^q \qquad \frac{1}{p}+\frac{1}{q}=1 \quad p,q \neq \infty$

$(L^1)^* = L^\infty$

$(L^\infty)^* \supsetneq L^1$

March 1 Zero set of real Analytic functions

• Taylor series expansion at every pt has a different radius but two points connected in compact set → finitely many balls



zero in this neighbourhood

real Analytic ⇒ non-zero radius of convergence

Then $\bigcup_{x \in A} B(x, \frac{r_x}{2}) \cap \gamma$ is open cover.

$\bigcup_{i=1}^{k} B(x_i, \frac{r_{x_i}}{2}) \cap \gamma$

finite sub ~~open~~ cover of γ

⇒ can find a finite subcover

$\boxed{r = \min\{r_i\}}$ constant.

finitely many → $|f| \geq k$ attained → ~~near~~ $|f| > 0$

globally 0

↳ wherever Taylor series

• Semi norm: They are exactly like semi norms except $p(x) = 0$ might not mean $x = 0$

$C^\infty(\mathbb{R}^n) \qquad p_k(f) = \|f\|_{C^k(B_k(0))}$

$$d(u,v) = \sum \frac{1}{2^k} \frac{p_k(u-v)}{1+p_k(u-v)}$$

$(X, \|\cdot\|)$ Base at 0. $\{x \in X \mid \|x\| < \varepsilon\}$

Basic open set β (collection of open sets): Given open set U, $x \in U$, $\exists\, B \in \beta$ s.t. $x \in B \subseteq U$

Idea: Basic open sets are

$\bigcup_{x \in B_1(u)}$ $B_1(u) \times$ can't find $B_1(u)$ in $B_{1/2}(u)$

$\bigcup B_{1/n}(u)$ $n \in N$ ✓

Base at 0 : $\{x \in X \mid \|x\| < \varepsilon\}_\varepsilon$

Basic open set

· Subbase: A collection S of open sets s.t. the collection of finite intersections of elements from S forms a base.

Norm → defines a base $\{\|x\| < \varepsilon \mid x \in X\}$

seminorm → defines a subbase.

X → linear topological space / vector space (Frechét spaces) metric from semi-norms.

$C^\infty(\mathbb{R}^n)$ seminorms

$S(\mathbb{R}^n)$ seminorms → full norms capture entire space

Cannot have a single norm which gives metric. Have to take sequence.

Recall $u \in S(\mathbb{R}^n)$

$$P_k(u) = \sup_{\substack{|\alpha|, |\beta| \leq k \\ x \in \mathbb{R}^n}} |x^\alpha \partial^\beta u(x)|$$

These are legitimate norms.

$$d(u, v) = \sum \frac{1}{2^n} \frac{P_k(u-v)}{1 + P_k(u-v)}$$

→ (Cont'd) A subbase is given by: $B_\varepsilon(0) \subseteq \mathbb{R}$

$$\{ P_k^{-1}(B_\varepsilon(u)) : k \in N, \varepsilon > 0 \}$$

Distribution: Dual space of C^∞, C_c^∞, S

· Characterization of cont linear functionals on X via the semi-norms.

Recall: Normed linear space $T \in X^*$

$|Tx| \leq c\|x\| \quad \forall x \in X$

cont. linear funcâl can be characterized by norm

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$T : X \longrightarrow \mathbb{C}$ cont.

$\downarrow$

Frechét space

when T is cont.

$\forall \varepsilon > 0$, one should be able to chose a basic open set B around 0, s.t. $T(B) \subseteq B_\varepsilon(0)$

We know ~~that~~ a typical basic open set in X

$$B = \bigcap p_{ij}^{-1}([0, \varepsilon_j])$$

$p_{i_1} \quad p_{i_2} \quad \dots$

Dont want to fin these

$x \in X$

$$p_{ij}\left(\frac{\varepsilon_j x}{2 p_{ij}(x)}\right) = \frac{\varepsilon_j}{2} \cancel{\frac{p_{ij}(x)}{p_{ij}(x)}}$$

for $T_B \subseteq B_\varepsilon$

$$\left| T\left(\frac{\varepsilon_j x}{2 p_{ij}(x)}\right) \right| < \varepsilon \iff |T(x)| < \frac{2\varepsilon}{\varepsilon_j} p_{ij}(x)$$

Finally: $T \in X^*$

$\Updownarrow$

$\exists$ constants $c, i_1, i_2, i_3 \dots i_n$

$$|T(x)| \leq C \max \{p_{i_1}(x), p_{i_2}(x), \dots$$

where $c = \max_j \left\{\frac{2\varepsilon}{\varepsilon_j}\right\}$

- derivative of Tempered distribution

dual of $S(\mathbb{R}^n) \rightarrow S^*(\mathbb{R}^n)$

$w \in S^*(\mathbb{R}^n)$

$u \in S(\mathbb{R}^n)$

$$(\partial_j w, u) \stackrel{def}{=} -(w, \partial_j u)$$

- Remark: Agrees with IBP (integr by parts) when $w, u \in S(\mathbb{R}^n)$

schwartz space $\subseteq$ Tempered

No boundary terms.

• eg Heaviside function

$$H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$H(x) \in S^*(\mathbb{R}^n)$

is in S^*: acts on $u \in S$
$$(H, u) = \int_{\mathbb{R}} H(x)\, u(x)\, dx$$
$$= \int_0^\infty u(x)\, dx < \infty$$

• Derivative:
$$(H', u) = -(H, u')$$
$$= -\int_0^\infty u'(x)\, dx$$
$$= u(0)$$
$$= (\delta_0, u)$$

• $H_1(x) = \begin{cases} x & x \geq 0 \\ 0 & x < 0 \end{cases}$

$H_1' = H$

March 5 Tue 3:30pm

• $X \to$ Frechet space
  $\hookrightarrow$ metric generated by countable seminorms

$T : X \xrightarrow{\text{cont}} \mathbb{C}$ $T \in X^*$ → get min

$\Updownarrow$

$\exists\, i_1, i_2, \ldots i_k$ st. $|Tu| \leq \max_j P_{i_j}(u)$

• <u>Fourier</u> Transform of a tempered distribution
$w \in S^*(\mathbb{R}^n)$

$(Fw, u) = (\hat{w}, u) \overset{\text{def}}{:=} (w, \hat{u})$ $u \in S(\mathbb{R}^n)$

similarly define F^*
$$(F^* w, u) = (w, F^* u)$$

<u>Fact</u>. $F^* F = F F^* = I$ on $S^*(\mathbb{R}^n)$

$\langle FF^* \omega, u \rangle \overset{\text{def}}{=} \langle F^* \omega, Fu \rangle$

$= \langle \omega, \underbrace{F^* F u}_{} \rangle \underset{\text{checked}}{=} \langle \omega, u \rangle$

Check: $L^{p} \subseteq S^* (\mathbb{R}^n)$

$\omega \in L^{p}$ $\quad \langle \omega, u \rangle = \int_{\mathbb{R}^n} \omega u \quad , u \in S(\mathbb{R}^n)$

$u_n \to u$ in $S(\mathbb{R}^n)$

$\int_{\mathbb{R}^n} \omega u_n \longrightarrow \int_{\mathbb{R}^n} \omega u$

$\left| \int_{\mathbb{R}^n} \omega (u_n - u) \right| \underset{\text{Holder}}{\leq} \| \omega \|_{L^p} \| u_n - u \|_{L^q}$

$u \in S(\mathbb{R}^n) \subseteq L^q(\mathbb{R}^n)$

$\int |u|^q = \int \frac{|u|^q |x|^{2m}}{|x|^{2m}} < \infty$

$u |x|^{2m/q} \in L^q$ $\quad \leq K^q \left(\int_{\mathbb{R}^n} \frac{1}{x^{2m}} \right.$

$\leq K$

choose m high enough so that integral exists

- Comment

☆ $\quad X \subseteq Y$ Banach / Topological vector space?

$\Downarrow$

$Y^* \subseteq X^*$

Check: One sufficient condition for ☆ to hold is
X dense in Y
X has a finer topology than the one induced by Y

linear functional cont on Y restrict to X.
cont on X also if topology on X is finer.

(If cont on Y $\Rightarrow$ ~~next~~ Cont on X as well)

$$X \overset{\text{embedding}}{\subseteq} Y$$
$$\downarrow$$
$$Y^* \subset X^*$$
embedding

if Y^* is embedding in X^*
one linear functional in Y does not have ~~to~~ two linear functionals from X.
restriction of Y^* to X^* should be unique.
$\therefore$ No two linear functionals on X should extend to the same Y!
For X dense in Y $\rightarrow$ extension is unique.

$$S(\mathbb{R}^n) \subseteq L^p$$
$$\Downarrow$$
$$L^q \subseteq S^*(\mathbb{R}^n)$$

To check:
$S(\mathbb{R}^n)$ dense in L^p
$S(\mathbb{R}^n)$ has a finer topology

finer: more open sets

- Examples of Tempered distributions

Principal Value
$$PV\left(\frac{1}{x}\right) \in S^*(\mathbb{R})$$

Define Action
$$\left(PV\left(\frac{1}{x}\right), u \right) \stackrel{def}{=} \lim_{\varepsilon \to 0} \int_{\mathbb{R}\setminus(-\varepsilon, \varepsilon)} \frac{u(x)}{x}\,dx \quad <_{\infty}^{\text{check}}$$

Cauchy Principal Value

$$\int \frac{u(x)}{x}\,dx = \lim_{a \to 0} \int_{-\infty}^{a} \cdots + \lim_{b \to 0} \int_{b}^{\infty} \cdots$$

$$-\infty + \infty$$

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}\setminus(-\varepsilon, \varepsilon)} \cdots$$

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Tuesday 3-5pm HW2

Also check: $u_n \to u$

$$(PV(\tfrac{1}{x}), u_n) \to (PV(\tfrac{1}{x}), u)$$

$$I(\varepsilon) = \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \frac{u(x)}{x}\,dx$$

$t = -x$

$$'' \int_\varepsilon^\infty \frac{u(x) - u(-x)}{x}\,dx$$

~~Hint~~ : MVT
or Cauchy

· Fourier Transform

$$(F\delta_0, \underset{\xi}{u}) \overset{\text{def}}{=} (\delta_0, \underset{x}{Fu})$$

$$= \left(\delta_0, \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(\xi) e^{-ix\xi}\, d\xi\right)$$

value of f at x = 0

$$= \frac{1}{(2\pi)^{n/2}} \int u(\xi)\, \cancel{e^{-i\xi 0}}\, d\xi \qquad (u \in S)$$

$$\boxed{F\delta_0 = \frac{1}{(2\pi)^{n/2}}}$$

$$(\partial_j c, u) = -(c, \partial_j u)$$

$$= -c \int_{\mathbb{R}^n} \partial_j u$$

$$'' = -c \; u\big|_{\partial\mathbb{R}^n} = 0$$

Inversion $F^* 1 = (2\pi)^{n/2} \delta_0$

• ~~Heaviside~~

Consider $H_\varepsilon(x) = \begin{cases} e^{-\varepsilon x} & x \geq 0 \\ 0 & x < 0 \end{cases}$

$$\left(\hat{H}_\varepsilon(\xi), u(\xi)\right) = \left(H_\varepsilon(x), \hat{u}(x)\right)$$

$$= \frac{1}{(2\pi)^{n/2}} \int_0^\infty e^{-\varepsilon x} \left(\int_{-\infty}^\infty u(\xi) e^{-i\xi x} d\xi \right) dx$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^\infty \left(\int_0^\infty e^{-\varepsilon x} u(\xi) e^{-i\xi x} dx \right) d\xi$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^\infty \left(u(\xi) \frac{e^{-x(\varepsilon + i\xi)}}{(-(\varepsilon+i\xi))} \Bigg|_0^\infty \right) d\xi$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^\infty \left\{ \frac{u(\xi)}{(\varepsilon + i\xi)} \right\} d\xi$$

Conclusion $\boxed{\hat{H}_\varepsilon(\xi) = \frac{1}{\varepsilon + i\xi}}$

Fact: ~~Note that~~ F, F^* are cont. on $S^*(\mathbb{R}^n)$

$w_n \to w \iff (w_n, u) \to (w, u) \quad \forall u \in S(\mathbb{R}^n)$

To check cont.: $w_n \to w \longrightarrow (w_n, Fu) \to (w, Fu)$

$\Updownarrow \qquad\qquad\qquad\qquad\qquad \Updownarrow$ def

$Fw_n \to Fw \leftarrow (Fw_n, u) \to (Fw, u)$

$\forall u \in S(\mathbb{R}^n)$

check: $H_\varepsilon(x) \xrightarrow{S^*(\mathbb{R}^n)} H(x)$ as $\varepsilon \to 0$

$\hat{H}_\varepsilon(\xi) \to \hat{H}(\xi)$

Conclusion: $\hat{H}(\xi) = \lim_{\varepsilon \to 0} \left(\frac{1}{\varepsilon + i\xi} \right)$

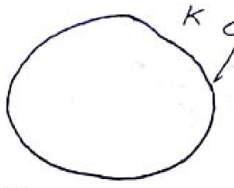
apply to u and ~~then~~ take limit

March 8   Digression Ω open in $\mathbb{R}^n$

Distribution : $\mathcal{D}(\Omega) := \left(C_c^\infty(\Omega)\right)^*$

Ref. Rauch's book Appendix.
Crash course in dist. theory.

Thm def


K ~~cpt~~ set

$w: C_c^\infty(\mathbb{R}^n) \to \mathbb{C}$
linear funct'al

$\left.\begin{array}{l} \varphi_n \in C_c^\infty(\mathbb{R}^n) \\ \quad\downarrow \quad \text{in all derivatives} \\ \varphi \in C_c^\infty(\mathbb{R}^n) \\ \underline{\operatorname{supp}\varphi_n, \operatorname{supp}\varphi \subseteq K} \end{array}\right\}$

$w \in \mathcal{D}'(\mathbb{R}^n) \iff w(\varphi_n) \to w(\varphi)$

Support of a distribut'n

$w \in \mathcal{D}'(\mathbb{R}^n)$ vanishes on Ω (open) $\subseteq \mathbb{R}^n$ if
$(w, u) = 0 \quad \forall u \in C_c^\infty(\Omega)$

$\operatorname{supp} w :=$ complement of "largest" vanishing set.

Eg: $\operatorname{supp} \delta_0 = \{0\}$

Take $\varphi \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$

$\delta_0(\varphi) = 0$

δ_0 vanishes outside $\{0\}$

Remark ~~Any~~ The above only proves $\operatorname{supp}\delta_0 \subseteq \{0\}$
convince yourself ~~that~~ $\operatorname{supp}\delta_0 \neq \phi$.

Next aim: to characterize (tempered) distributions with cmpct support.

Let $\left(C^\infty(\mathbb{R}^n)\right)^* := \mathcal{E}'(\mathbb{R}^n)$

Thm $\quad \mathcal{E}'(\mathbb{R}^n) =$ distributions with cmpct support.

Proof Suppose $w \in \mathcal{D}'(\mathbb{R}^n)$ and $\operatorname{supp} w$ is cmpct $= K$

Arbitrary $f \in C^\infty(\mathbb{R}^n)$

Take a smooth cut-off $\chi \in C_c^\infty(\mathbb{R}^n)$ s.t. $\chi \equiv 1$ in a nbhd around K.

$(w, f) := (w, \chi f)$

dist with cmpt supp $\subseteq \mathcal{E}'(\mathbb{R}^n)$

Check that ω is cont on $C^\infty(\mathbb{R}^n)$

For ~~⊇~~ the other inclusion:

Choose $\omega \in (C^\infty(\mathbb{R}^n))^* = \mathcal{E}'(\mathbb{R}^n)$

$\exists k$

$$|\langle \omega, u\rangle| \leq c\, p_k(u)$$

when k increases norm increases. Increasing sequence of seminorms. max of finitely many = last element.

$$= c\, \|u\|_{C^k(\overline{B_k(0)})}$$

$\operatorname{supp} \omega \subseteq \overline{B_k(0)}$

~~Thm~~ Heuristic: Fourier Transform exchanges smoothness with decay at infinity.

Thm: Let $\omega \in \mathcal{E}'(\mathbb{R}^n)$. Then we have ~~is a~~

(a) $\hat{\omega}$ is a smooth function (representative)

(b) $\hat{\omega}(\xi) = \frac{1}{(2\pi)^{n/2}} \langle \omega, e^{-ix\xi}\rangle$ (pointwise characterisation)

(c) $\hat{\omega}$ has holomorphic extension to $\xi \in \mathbb{C}^n$ (Look up Paley Weiner)

Proof (c) easy to check:

$$\partial_{\xi_1} \hat{\omega}(\xi) = \lim_{h\to 0} \frac{\hat{\omega}(\xi+h) - \hat{\omega}(\xi)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\langle \omega, e^{-ix(\xi+h)}\rangle - \langle \omega, e^{-ix\xi}\rangle\right)$$

$$= \lim_{h\to 0} \left\langle \omega, \frac{e^{-ix(\xi+h)} - e^{-ix\xi}}{h}\right\rangle \quad \text{lin. oper}$$

using continuity, push limit inside, and then check Cauchy-Riemann eqns.

Cauchy Riemann for holomorphic funct of several var.

$f(z_1, z_2, \ldots, z_n) : \mathbb{C}^n \to \mathbb{C}^n$

$z_i = x_i + i y_i \qquad f = u + iv$

$$\left.\begin{array}{l} \partial_{x_i} u = \partial_{y_i} v \\ \partial_{y_i} u = -\partial_{x_i} v \end{array}\right\} \forall i.$$

complex analytic: if two Fourier transforms agree on some non-trivial set then they agree everywhere

(c) + (b): $\langle \hat{w}, u \rangle \stackrel{def}{=} \langle w, \hat{u} \rangle$

claim
$$\langle w, \hat{u} \rangle = \frac{1}{(2\pi)^{n/2}} \left[\int_{\mathbb{R}^n} u(\xi) \langle w, e^{-ix\xi} \rangle d\xi \right]$$

def

// ?

$$w\left(\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(\xi) e^{-ix\xi} d\xi \right)$$

w can't be pushed in : finite sum.
w: linear functional

Take Riemann sums finite and then take $\lim_{n\to\infty}$

$w($ lim   conv. in Schwartz

March 12 Tue 3:30pm

Recall: we were proving for $w \in \mathcal{E}'(\mathbb{R}^n)$

$\hat{w}$ is smooth and

$$\hat{w}(\xi) = \frac{1}{(2\pi)^{n/2}} \langle w, e^{-ix\xi} \rangle$$

Extension to $\mathbb{C}^n$: HW 4.

Main claim: $\langle w, \int u(\xi) e^{-ix\xi} d\xi \rangle = \langle u, \langle w, e^{-ix\cdot\xi} \rangle \rangle$

For notational convenience, we will write down the proof for $n=1$ ie on $\mathbb{R}$.

By a density argument, it is enough to consider $u \in C_c^\infty(\mathbb{R})$

$\operatorname{supp} u \subseteq [a, b]$

$$v(x) = \int_a^b u(\xi) e^{-i\xi\cdot x} d\xi \quad \underset{\text{Riemann sum}}{\longleftrightarrow} \quad v_k(x) = \frac{b-a}{k} \sum_{r=1}^{k-1} (a + rh) e^{-ix\cdot(a+rh)}$$

Suppose $v_k(x) \longrightarrow v(x)$ in $C^\infty(\mathbb{R}^n)$

$$\text{so, } \left\langle w, \frac{b-a}{k} \sum_{r=1}^{k-1} u(a+rh) e^{-ix\cdot(a+rh)} \right\rangle$$

$\langle w, v \rangle \leftarrow$ converges to   since * $w \in (C^\infty(\mathbb{R}^n))'$

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ω is indual of $C^\infty(\mathbb{R}^n)$

$\hookrightarrow$ seminorms

~~So~~ ~~(ω~~ In n-dimensions just $a_1 + rh_1, a_2 + rh_2 \ldots$ n summations.

still finite $\to$ (needed to push ω inside)

$$\frac{b-a}{k} \sum_{r=1}^{k-1} u(a+rh)\langle \omega, e^{-ix(a+rh)} \rangle$$

$\downarrow$ Rimann Integral definition

$$\langle \omega, v \rangle \longleftrightarrow \int_a^b u(\xi)\langle \omega, e^{-ix\xi} \rangle \, d\xi$$

Need to check ~~that~~

$\Delta_x^\alpha v_k(x) \longrightarrow \Delta_x^\alpha v(x) \;\forall$ multiindices

diff $\to$ multiply ξ such α is same.

<u>Extension of Liouville Thm</u>

<u>Thm</u> If $\omega \in S^*(\mathbb{R}^n)$ is supported ~~to be~~ at $\{0\}$, $\exists\, k$ and $a_\alpha \in \mathbb{C}$, st.

$$\omega = \sum_{|\alpha| \le k} a_\alpha \Delta^\alpha \delta_0$$

part of HW-4
Many Books
Taylor, Rauche ..

$\hookrightarrow$ Suppose $u \in S^*(\mathbb{R}^n)$ and $\Delta u = 0$.

we have defined derivatives for tempered dist. and add^n.

Then u is a polynomial in $x_1, x_2 \ldots x_n$.

<u>Proof</u> $\Delta u = 0 \xrightarrow{F.T.} -|\xi|^2 \hat{u} = 0$ ① (check!)

$$[\hat{\Delta u} = -|\xi|^2 \hat{u}]$$

$$\langle \hat{\Delta u}, \varphi \rangle \stackrel{def}{=} \langle \Delta u, \hat{\varphi} \rangle \stackrel{def}{=} \langle u, \Delta \hat{\varphi} \rangle$$

$$\langle u, |\xi|^2 \hat{\varphi} \rangle$$

then ① $\to \hat{u} = 0$ or $\xi = 0$

$\therefore \hat{u} \ne 0$ iff $\xi = 0$ $\boxed{\operatorname{supp} \hat{u} = \{0\}}$

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$$\hat{u} = \sum_{|\alpha| \le k} a_\alpha \, \Delta^\alpha \delta_0$$ By previous thm

$$u = F^*(\hat{u}) = \sum_{|\alpha| \le k} a_\alpha |\xi|^\alpha (2\pi)^{n/2}$$

$$F^* \delta_0 = (2\pi)^{n/2}$$

March 19 Tue 3:30-5pm

Presentⁿ: Friday April 12th 3:30-5:30pm 2 ppts.
Saturday April 13th 9:00am-1pm 3 ppts

Fundamental solns:

Thm: Laplacian:

$n \ge 3$ $\quad \Delta(|x|^{2-n}) = C_n \delta_0$ on $\mathbb{R}^n$

$$C_n = -(n-2)|S^{n-1}|$$

$n=2$ $\quad \Delta(\log|x|) = C_2 \delta_0$ $\quad C_2 = 2\pi$

$(\partial_t - \Delta)$: $\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}$ Heat Kernel

Proof GGI: $\displaystyle\int_\Omega \Delta u v - u \Delta v = \int_{\partial\Omega} v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n}$

Choose $u \in C_c^\infty(\mathbb{R}^n)$ $\quad v = |x|^{2-n}$ $\quad \underset{\Omega_\varepsilon}{\Omega} = \mathbb{R}^n \setminus B_\varepsilon(0)$

$$(\Delta u, |x|^{2-n}) = \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \Delta u \, |x|^{2-n}$$

$u(\Delta |x|^{2-n})$ operating on u

$$= \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \Delta u |x|^{2-n} - u \Delta |x|^{2-n}$$

to for $x \neq 0$

Check that $\Delta \left(\frac{1}{(x_1^2 + \cdots x_n^2)^{\frac{n-2}{2}}} \right) = 0$

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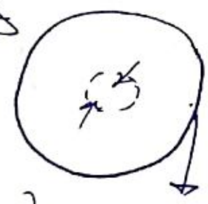
Green's 2nd thm $\lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon} \cancel{\phi}\left[\varepsilon^{2-n} \frac{\partial u}{\partial r} + (2-n)\,\varepsilon^{1-n} u\right]$

goes to 0

$= -(n-2)\,|S^{n-1}|\, u \cos$



$\varepsilon^{1-n} \lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon} u \cos \to u \cos \varepsilon^{n-1} |S^{n-1}|$

$|\partial B_\varepsilon| = \varepsilon^{n-1} |S^{n-1}|$

$\frac{\partial u}{\partial r} \le M$ on $B(g,2)$

$\varepsilon^{2-n} \times \varepsilon^{n-1} |S^{n-1}|$

Boundary value problem for upper half space

$$\left(\frac{\partial^2}{\partial y^2} + \Delta\right) u = 0 \qquad y > 0 \quad x \in \mathbb{R}^n$$

$$u\big|_{\{0\}\times\mathbb{R}^n} = f(x). \quad \in S^*(\mathbb{R}^n)$$

FT

$$\left.\begin{aligned}\left(\frac{\partial^2}{\partial y^2} - |\xi|^2\right)\hat{u} &= 0 \\ \hat{u}(0,\xi) &= \hat{f}(\xi)\end{aligned}\right\} \text{ODE}$$

Soln.

$$\hat{u}(y,\xi) = C_0(\xi)\, e^{y|\xi|} + C_1(\xi)\, e^{-y|\xi|}$$

we want $\hat{u} \in S^*$

So we set $C_0(\xi) = 0$

then $\hat{u}(y,\xi) = C_1(\xi)\, e^{-|y|\xi}$

B.C.

$$\boxed{\hat{u}(y,\xi) = e^{-y|\xi|} \hat{f}(\xi)}$$

inverse fourier of product is convol^n. of individual IFT.

IFT of $e^{-y|\xi|} = ?$

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We are looking for fundamental sol^n in the sense that $f(x) = \delta_0$

Fundamental soln: $= P(y, x)$

$$\hat{P}(y,\xi) = e^{-y|\xi|} \frac{1}{(2\pi)^{n/2}}$$ (I.F.T of $\delta_0 \to$ const)

for $n=1$

$$P(y,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-y|\xi|} e^{ix\xi}\, d\xi$$

$$= \frac{1}{2\pi}\int_0^{\infty} e^{-y\xi} \;\; + \frac{1}{2\pi}\int_0^{\infty} e^{-y\xi} e^{-ix\xi}\, d\xi$$

change of var.

$$= \frac{1}{2\pi} \frac{e^{\xi(-y+ix)}}{(-y+ix)}\Big|_0^{\infty} + \frac{1}{2\pi} \frac{e^{-\xi(y+ix)}}{-(y+ix)}\Big|_0^{\infty}$$

$$= \frac{1}{\pi}\frac{y}{y^2+x^2}$$

Taylor

Subordinate Identity

$$e^{-yA} = \frac{y}{\sqrt{2\pi}}\int_0^{\infty} e^{-y^2/4t} e^{-tA^2}\, t^{-3/2}\, dt$$

$A>0, y>0$

$$P(y,x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-y|\xi| + ix\cdot\xi}\, d\xi$$

$$= \frac{1}{(2\pi)^n}\frac{y}{\sqrt{2\pi}}\int_{\mathbb{R}^n}\int_0^{\infty} e^{-y^2/4t} e^{-t|\xi|^2 + ix\cdot\xi}\, t^{-3/2}\, dt\, d\xi$$

$$= \frac{1}{(2\pi)^n} \frac{y}{\sqrt{2\pi}} \int_0^\infty e^{-(x^2+y^2)/4t}\, t^{-n/2}\, t^{-3/2}\, dt$$

+ x, y const wrt t.

$z = \frac{x^2+y^2}{4t}$ } Gamma Integral

$dz = \frac{(x^2+y^2)}{4t^2}(-1)\,dt$

$$= c_n \frac{y}{(y^2 + |x|^2)^{\frac{n+1}{2}}}$$

$$c_n = (\pi)^{-\frac{n+1}{2}}\, \Gamma\left(\frac{n+1}{2}\right)$$

Wave Eq^n:

$$\frac{\partial^2 u}{\partial t^2} = \Delta u$$

$u(0, x) = f(x)$

$u_t(0, x) = g(x)$

Assume $f, g \in S^*(\mathbb{R}^n)$ and we look for sol^{ns} $u(t, x)$ s.t. $u(t, \cdot) \in S^*(\mathbb{R})$

$$\frac{\partial^2 \hat{u}}{\partial t^2} + |\xi|^2 \hat{u} = 0$$

$$\hat{u}(0, \xi) = \hat{f}(\xi)$$

$$\hat{u}_t(0, \xi) = \hat{g}(\xi)$$

Solving $\hat{u}(t, \xi) = \hat{g}(\xi)\, |\xi|^{-1} \sin(t|\xi|) + \hat{f}(\xi) \times (1) \cos(t|\xi|)$

Fundamental soln: $R(t, x)$

Let $f = 0$, $g = \delta_0$

$$\hat{R}(t, \xi) = \frac{1}{(2\pi)^{n/2}} |\xi|^{-1} \sin(t|\xi|)$$

$\partial_t^2 + \partial_{x_1}^2 + \dots \partial_{x_n}^2$

$\left(\frac{\partial^2}{\partial (iy)^2} + \Delta\right) u = 0$ Main idea. Can we replace y by iy?

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We know $F^*(e^{-y|\xi|}) = c_n \dfrac{y}{(y^2+|x|^2)^{\frac{n+1}{2}}}$ ①

Check

$$F^*\left(\frac{e^{-y|\xi|}}{|\xi|}\right) = d_n(y^2+|x|^2)^{-\left(\frac{n-1}{2}\right)} \quad \forall y \geq 0$$

(integrate above ① w.r.t. y) → Holds only for $Re(y) \geq 0$ really

For $y \in \mathbb{C}$

$\dfrac{e^{-y|\xi|}}{|\xi|}$ is holomorphic in y

When $Re\, y \geq 0$, $\dfrac{e^{-y|\xi|}}{|\xi|} \in S^*_\xi(\mathbb{R}^n)$

$F^*\left(\dfrac{e^{-y|\xi|}}{|\xi|}\right)$ makes sense as a tempered distribution

Check : $F^*\left(\dfrac{e^{-y|\xi|}}{|\xi|}\right)$ is holomorphic in y

(Use Cauchy Riemann)

$$F^*\left(\frac{e^{-y|\xi|}}{|\xi|}\right) \overset{\#}{=} d_n(y^2+|x|^2)^{-\left(\frac{n-1}{2}\right)} \quad Re(y)>0$$

If we can prove $\dfrac{e^{-y_n|\xi|}}{|\xi|} \xrightarrow{S^*(\mathbb{R}^n)} \dfrac{e^{-y|\xi|}}{|\xi|}$ $\quad Re\, y_n > 0$, $Re\, y = 0$

By continuity of F^* on $S^*(\mathbb{R}^n)$

If $y = it + \varepsilon$

$$R(t,x) = \lim_{\varepsilon \to 0} Im\, (|x|^2 - (t-i\varepsilon)^2)^{-\frac{n-1}{2}}$$

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March 22
Friday 3:30-5

Recall: we established fundamental soln of wave Eqn

Next Fri Problem HW-3 4

$$(\partial_t^2 - \Delta)u = 0$$

$$u(0,x) = 0$$

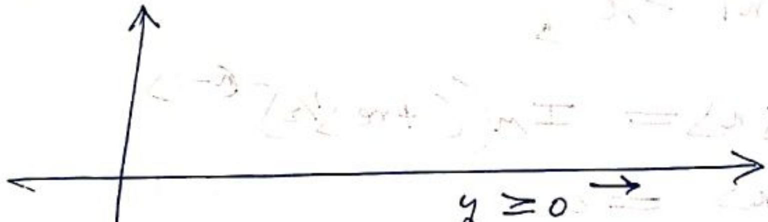
$$u_t(0,x) = \delta_0 \quad \text{is}$$

$$R(t,x) = \lim_{\varepsilon \to 0} c_n' \,\mathrm{Im}\left(|x|^2 - (t - i\varepsilon)^2\right)^{-\frac{n-1}{2}}$$

The idea: $(\partial_t^2 + \Delta)u = 0$

If we replace t with ~~it~~ it, we get wave Eqn

Calculated $F^* \cancel{e^{-\partial}} \left(\dfrac{e^{-y|\xi|}}{|\xi|} \right)$ for $y \geq 0$



would like to replace y by iy. This is done by analytically continuing to $\mathrm{Re}\, y > 0$ and then taking limit as $\mathrm{Re}\, y \to 0$.

$y \mapsto F^*\left(\dfrac{e^{-y|\xi|}}{|\xi|} \right)$ is a $S^*(\mathbb{R}^n)$-valued holomorphic analytic function.

Overview
Def
not precise

X: Topological vec. space

$f: \underset{\substack{\text{domain}\\ \text{in } \mathbb{C}}}{\Omega} \to X$. f is strongly holomorphic if $\lim_{h \to 0} \dfrac{f(z+h) - f(z)}{h}$ exists (in the top of X) $\forall z \in \Omega$

Most well-known theorems about $\Omega \to \mathbb{C}$ holomorphic theory goes over in this setting as well

Ref: Rudin Functional Analysis ~~this~~

Weak holomorphicity:

Def $f: \Omega \to X$ is "weakly" holomorphic if Tf is holomorphic $\forall\, T \in X^*$

T: linear functional.

strongly holomorphic $\Rightarrow$ weakly holomorphic

Thm If X is Frechet, converse is also true.
(Rudin Functional Analysis Thm 3.31)

$$R(t,x) = \lim_{\varepsilon \to 0} C_n' \operatorname{Im}\left(|x|^2 - (t - i\varepsilon)^2\right)^{-\frac{n-1}{2}} \longrightarrow \text{singularity at } |x| = t$$

Consequences: If $|x| > t$,

$$R(t,x) = \operatorname{Im}\left((+ve)^{1/2}\right)^{-(n-1)}$$

$$R(t,x) \equiv 0$$

finite speed of propagation of fundamental solution.

Basically start with an initial condition (disturbance) supported at the origin then the f.s. (fundamental soln) at time t is supported within $B_t(0)$ (Ball of radius t)

If n is odd, $\frac{n-1}{2} \in \mathbb{N}$, $R(t,x) \equiv 0$ even when $|x| < t$

$|x| = t$ only nonzero for $n = $ odd

So $\text{supp}(R(t,x)) \subseteq \{x \in \mathbb{R}^n \quad |x| = t\}$

This is known in physics literature as sharp Huygen's Principle.

In dim $n=3$ $\quad R(t,x) = \delta(|x| - t)$

$$R(t,x) = \lim_{\varepsilon \to 0} C_3' \, \text{Im} \frac{1}{|x|^2 - (t - i\varepsilon)^2}$$

$$\hat{H}(\xi) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon + i\xi}$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon - i\xi} = 2\pi \, \delta_0 - \hat{H}(\xi)$$

$$\Big[\; H(-x) = 1 - H(x)$$

$$\hat{H}(-\xi) = (2\pi)^{1/2} \delta_0 - \hat{H}(\xi) \; \Big]$$

$$\sqrt{2\pi} \, \delta_0 = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon + i\xi} + \lim_{\varepsilon \to 0} \frac{1}{\varepsilon - i\xi}$$

Plemelji Jump Relation.

Fundamental Solutions

Laplacian – Want to

$$\Delta u = f \qquad f, u \in S^*(\mathbb{R}^n).$$

$$\Delta\left(|x|^{2-n}\right) = \delta_0.$$

Convolution of a ~~Tempered~~ distribution with a func^n.

$$\omega \in \mathcal{D}(\mathbb{R}^n), \; \varphi, u \in C_c^\infty(\mathbb{R}^n)$$

$$(\omega * \varphi, u) := (\omega, u * R\varphi)$$

$$R\varphi(x) := \varphi(-x) \qquad (R: \text{Reflection})$$

Verifiable when $\omega, u, \varphi \in C_c^\infty(\mathbb{R}^n)$

First Compute

$\langle \delta_0 * u, \varphi \rangle$

$\overset{def}{=} \langle \delta_0, \varphi * Ru \rangle$

$= \delta_0\left(\int \varphi(x-y) Ru(y) dy \right)$ $\qquad \delta_0(f(x)) = f(0)$

$= \int \varphi(-y) u(-y) dy$

$= \int \varphi(y) u(y) dy$

$\boxed{\delta * u = u}$

~~Heat~~ Laplacian : $\Delta u = f$

$\Delta(|x|^{2-n}) = \delta_0$

$\Delta(|x|^{2-n}) * f = f$

$\shortparallel$

$\Delta(|x|^{2-n} * f) = f$

Heat Eq^n : The heat ~~eq^n~~ kernel $p(t,x) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}$ is a fundamental solution for $\partial_t - \Delta$

Wave Eq^n $\quad (\partial_t^2 - \Delta) u = 0$

$u(0,x) = f(x)$

$u_t(0,x) = g(x)$

$u(t,x) = R(t,x) * g + \partial_t R(t,\cdot) * f$

Corollary $\text{supp}(\omega * \varphi) \subseteq \overline{\text{supp}(\omega) + \text{supp}\,\varphi}$

If $f, g \in C_c^\infty(\mathbb{R}^n)$, then $u(t,x)$ has finite speed of propagation.

Spectral theorem for unbounded self adjoint operator

$H \longrightarrow$ separable Hilbert space.

$T : H \longrightarrow H$ self-adjoint unbounded operator

Ref Taylor Vol II chap 8 section 2

Statement : $\Omega \longrightarrow$ locally compact T_2 (check) ↳ Hausdorff

$\exists\ u : L^2(\Omega) \longrightarrow H$ unitary

s.t. $u^{-1} T u f(x) = a(x) f(x) \quad \forall f \in L^2(\Omega)$

(Infinite dimensional analogue of diagonalization)

Any unbounded self-adjoint operator is unitarily equivalent to a multiplication operator

Same thing as symmetric matrix is diagonalizable

Example $\quad T = \Delta \qquad H = L^2(\mathbb{R}^n)$

u = Inverse Fourier Transform

$$F \Delta F^{-1} \underbrace{\hat{f}(\xi)}_{f(x)} = -|\xi|^2 \hat{f}(\xi)$$

$\widehat{\Delta f(x)}$

Spectral theorem allows us to define "functions" φ of T. for sufficiently nice φ
↳ bounded, cont, decaying.

En $\quad u^{-1}\varphi(T)u\, f(x) \stackrel{def}{=} \varphi(a(x))\, f(x)$

$$\widehat{\varphi(\Delta)f}(\xi) = \varphi(-|\xi|^2)$$

$$\widehat{(1-\Delta)^{1/2}f}(\xi) = (1+|\xi|^2)^{1/2}\hat{f}(\xi)$$

March 25 Monday 5–8pm

spectral thm

H : separable Hilbert space

T : self adjoint unbounded operator.

$\exists$ finite measure space (Ω, μ) and

$u : L^2(\Omega) \longrightarrow H$ unitary map

s.t. $\forall f \in L^2(\Omega)$

$$u^{-1}\, T\, u\, f(x) = \underset{\text{real valued measurable function}}{a(x)}\, f(x)$$

Basically: any unbounded self adjoint operator is "unitarily equivalent" to a multiplication operator

The spectral theorem allows us to define "functions of unbounded self adjoint operators"

φ : Borel function on $\mathbb{R} \to \mathbb{C}$

We want to define $\varphi(T)$

if T were symmetric matrix, diagonalizable

$$P^{-1}TP = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

define $\varphi(T)$ as

$$P^{-1}\varphi(T)P = \begin{pmatrix} \varphi(\lambda_1) & & \\ & \ddots & \\ & & \varphi(\lambda_n) \end{pmatrix}$$

$\hookrightarrow$ This is self adj.

for $U = F^{-1}$ (Inv. Fourier T)

$$F\Delta F^{-1}\hat{f}(\xi) = -|\xi|^2\hat{f}(\xi)$$

Observe that we defined $e^{t\Delta}$, $e^{it\sqrt{-\Delta}}$ adhoc

$$\varphi_1(x) = e^{tx} \qquad \varphi_2(x) = e^{it\sqrt{-x}}$$

Recall In dim $n=3$

$$R(t,x) = \lim_{\varepsilon \to 0} C_3 \operatorname{Im} \frac{1}{|x|^2 - (t - i\varepsilon)^2}$$

Want to Prove $R(t,x) = \delta(|x| - |t|)$

Case 1. $t>0$, $t<0$ will be similar.

$$R(t,x) = \lim_{\varepsilon \to 0} \operatorname{Im} \frac{1}{(|x|+|t|-i\varepsilon)(|x|-|t|+i\varepsilon)}$$

$$= \lim_{\varepsilon \to 0} \operatorname{Im}\left[\frac{1}{|x|+|t|-i\varepsilon}\right]\left[\frac{1}{|x|-|t|+i\varepsilon}\right]$$

never singular Re Part >0 — can be singular.

$$= \frac{1}{(|x|+|t|)} \lim_{\varepsilon \to 0} \operatorname{Im} \frac{1}{|x|-|t|+i\varepsilon}$$

$$= \frac{1}{|x|+|t|} \lim_{\varepsilon \to 0} \operatorname{Im} \frac{|x|-|t|-i\varepsilon}{(|x|-|t|+i\varepsilon)(|x|-|t|-i\varepsilon)}$$

$$= \frac{1}{|x|+|t|} \lim_{\varepsilon \to 0} -\frac{\varepsilon}{(|x|-|t|+i\varepsilon)(|x|-|t|-i\varepsilon)}$$

$$= \frac{-1}{|x|+|t|} \lim_{\varepsilon \to 0} \frac{1}{2i} \left[\frac{1}{|x|-|t|-i\varepsilon} - \frac{1}{|x|-|t|+i\varepsilon} \right.$$

$$= \frac{\mp 1}{2i(|x|+|t|)} \delta_{|x|-|t|}$$

$$\hat{H}(\xi) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon + i\xi}$$

$$H(x) = 1 - H(-x)$$

$$\hat{H}(\xi) = (2\pi)^{1/2} \delta_0 - \hat{H}(-x)$$

$$= (2\pi)^{1/2} \delta_0 - \lim_{\varepsilon \to 0} \frac{1}{\varepsilon - i\xi}$$

Plemelj $= (2\pi)^{1/2} \delta_0 - \lim \frac{1}{\varepsilon - \xi}$

$$(2\pi)^{1/2} \delta_0 = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon + i\xi} + \lim_{\varepsilon \to 0} \frac{1}{\varepsilon - i\xi}$$

$$(2\pi)^{1/2} \delta_0 = -i \left[\lim_{\varepsilon \to 0} \frac{1}{\xi - i\varepsilon} - \lim_{\varepsilon \to 0} \frac{1}{\xi + i\varepsilon} \right]$$

L^2 lies in tempered dist. We know how to differentiate Tempered dist.

~~Soto~~

Sobolev Spaces

Main ref Folland.

Basically, this will turn out to be one of the nicest settings for studying elliptic PDEs

Def $W^{k,2} = H^k(\mathbb{R}^n) = \{ f \in L^2(\mathbb{R}^n) \mid \underbrace{\partial^\alpha f}_{\text{in the sense of distrib}^n} \in L^2$
$\forall |\alpha| \le k \in \mathbb{N} \cup \{0\} \}$

$\partial^\alpha f \in L^2 \Leftrightarrow$

$\exists g \in L^2$ $(\partial^\alpha f, \varphi) = (g, \varphi) \ \forall \ \varphi \in C_c^\infty(\mathbb{R}^n)$

Fourier characterization.

Proposition: $f \in H^k(\mathbb{R}^n) \Leftrightarrow \cancel{(1+|\xi|^2)^{k/2}} (1+|\xi|^2)^{k/2} \hat{f}(\xi) \in L^2(\mathbb{R}^n)$

$(\xi^\alpha \hat{f}(\xi) = \widehat{\partial^\alpha f}(\xi) \in L^2$ $\forall |\alpha| \le k$ expand $(1+|\xi|^2)^{k/2}$ as polynomial in $|\xi|^2$).

Also $\|f\|_1^2 = \sum_{|\alpha| \le k} \|\partial^\alpha f\|_{L^2}^2$

$\sum \|\partial^\alpha f\|_{L^2} = \sum \|\xi^\alpha \hat{f}\|_{L^2}$

and $\|f\|_2^2 = \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1+|\xi|^2)^k d\xi$

are equivalent and both can be used to define $\|f\|_{H^k}^2$

prove: beyond a compact set leading term dominates

$\sum_{|\alpha| \le k} c |\xi|^\alpha \sim (1+|\xi|^2)^{k/2}$

$c_1 \| \ \ \|_1 \le \| \ \ \|_2 \le c_2 \| \ \ \|_1$

$\frac{1}{2}$ $\qquad$ 2

$\|f\|_{H^k} \stackrel{\text{prop}^n}{=} \|(1+|\xi|^2)^{k/2} \hat{f}(\xi)\|_{L^2}$

This allows us to extend the def^n of Sobolev spaces to $s \in \mathbb{R}$

$H^s(\mathbb{R}^n) = \{ u \in S^*(\mathbb{R}^n) \mid \hat{u}(\xi) \in L^1_{loc}(\mathbb{R}^n)$

and $\hat{u}(\xi)(1+|\xi|^2)^{s/2} \in L^2(\mathbb{R}^n) \}$

Trivial Remark: $H^{|s|} \subseteq L^2 \subseteq H^{-|s|}$

$s>0$ $\hat{u}(1+|\xi|^2)^{s/2} \in L^2 \to \hat{u} \in L^2 \to u \in L^2$

$s<0$ even if $\hat{u} \notin L^2$ $\hat{u}(1+\xi)^{-|s|/2}$ may make it L^2.

$\therefore u \in H^{-|s|} \not\to L^2$

Fact: $H^s(\mathbb{R}^n)$ is a Hilbert Space

$$(u,v) = \int_{\mathbb{R}^n} \hat{u}\,\overline{\hat{v}}\,(1+|\xi|^2)^s \, d\xi$$

$$Id : H^s(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n, \mu)$$

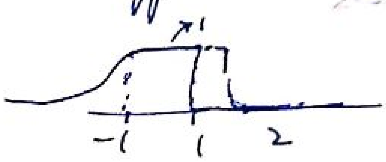
$$d\mu = (1+|\xi|^2)^s \, d\xi$$

This is a unitary operator. So H^s is Hilbert.

Propos^n: $C_c^\infty(\mathbb{R}^n)$ dense in $H^s(\mathbb{R}^n)$ (and as a biproduct, $S(\mathbb{R}^n)$ dense in $H^s(\mathbb{R}^n)$)

$\varphi_j (\varphi_\epsilon * f) \to f$ $\quad L^p \to L^p$

cuttoff $\quad \epsilon \to \frac{1}{j}$



Sobolev Embedding

Trace

Compact Sobolev Embedding

Ex) $n=1$ $f(x) = \frac{\sin(x)}{x}$

claim: $f \in H^s \ \forall s \in \mathbb{R}$

$f(x) = F^{-1}(\chi_{[-1,1]})$ (midterm question)

2) $\delta_0 \in H^s \iff s < -\frac{n}{2}$ [In particular $\delta_0 \notin L^2$]

$$\|\delta_0\|_{H^s}^2 = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (1+|\xi|^2)^s \, d\xi$$

$$= \int_{0}^{\infty} (1+r^2)^s r^{n-1} \, dr \, d\omega$$

$$\sim \left(\int_{B(1,0)} d\omega\right) \int_0^\infty (1+r^2)^s r^{n-1} \, dr$$

no problem at 0 but $r=\infty$ converges for $s<0$ to pull down $n-1$

$$\lesssim \int_0^5 (\ ) + \int_5^\infty \frac{1}{(r^2)^{-s}} r^{n-1}$$

$$\int_5^\infty r^{2s+n-1} \, dr$$

converges ~~at~~ for $2s+n-1 > -1$

$s < -\frac{n}{2}$

3) $\Delta \delta_0 \in H^s$ for what s?

Duality of H^s and H^{-s}

want to show: $(H^s)^* = H^{-s}$

Proof: Let $f \in H^s$, $g \in H^{-s}$. We want to see g as a member of $(H^s)^*$

linear continuous func^al on H^s.

Define $T_g : H^s \to \mathbb{C}$

$$T_g(f) = \int_{\mathbb{R}^n} \hat{f}\, \overline{\hat{g}}$$

multiply divide:

$$= \int_{\mathbb{R}^n} \hat{f}(\xi)(1+|\xi|^2)^{s/2}\, \overline{\hat{g}}(\xi)(1+|\xi|^2)^{-s/2} d\xi$$

Holder

$$\leq \|f\|_{H^s} \|g\|_{H^{-s}}$$

So, T_g is well defined and bounded and

$$\|T_g\| \leq \|g\|_{H^{-s}}$$

Claim: $g \mapsto T_g$ is injective.

Let $T_g = 0$ for some $g \in H^{-s}$

$$\int_{\mathbb{R}^n} \hat{\varphi}\, \overline{\hat{g}} = 0 \quad \forall \varphi \in H^s(\mathbb{R}^n)$$

$F. \; S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ is bijective. ($\subseteq$ dense $H^s(\mathbb{R}^n)$)

$$\int_{\mathbb{R}^n} \psi\, \overline{\hat{g}} = 0 \quad \forall \psi \in C_c^\infty(\mathbb{R}^n)$$

$$\Downarrow$$

$$\hat{g} = 0 \to g = 0$$

Claim: $g \mapsto T_g$ is surjective

Pick $T \in (H^s)^*$ We want to cook up g s.t. $T = T_g$

By Riesz Representation Thm

$$T(f) = (f, g')_{H^s} \quad \text{fixed } g' \in H^s$$

Scanned by CamScanner

Take $\hat{h} = \hat{g'} (1+|\xi|^2)^s$

$\langle f, g' \rangle_{H^s} = \langle f, h \rangle_{L^2}$ —

$$\langle f, g' \rangle_{H^s} = \int_{\mathbb{R}^n} \hat{f}(\xi) \, \overline{\hat{g'}(\xi)} \, (1+|\xi|^2)^s \, d\xi$$

$$= \int_{\mathbb{R}^n} \hat{f} \, \overline{\hat{h}}$$

Left to prove : $\hat{h}' \in H^{-s}$

$\therefore$ Bijective.

Riesz Represent^n also gives norm equality.

March 26 Tuesday 3:30 – 5pm

Comment : $L^2(\mu)$ is a Hilbert space for any positive measure μ. (Rudin : Real and Complex Analysis Ch4)

Two Banach spaces are equi if ~~they~~ are isometrically isomorphic.

Only Bijec^n doesn't help. contrac^n may distort measures in diff direc^{n}s $\frac{1}{c} \|\cdot\| < \|\cdot\| < c\|\cdot\|$

Sobolev Embedding Theorem

If $s > \frac{n}{2} + k$ then $H^s(\mathbb{R}^n) \overset{i}{\hookrightarrow} C^k(\mathbb{R}^n)$

and the embedding is continuous

ie. $\|f\|_{C^k(\mathbb{R}^n)} \leq C \|f\|_{H^s(\mathbb{R}^n)}$

Corollary : $\bigcap_{s \in \mathbb{R}} H^s \subseteq C^\infty(\mathbb{R}^n)$

Proof : By Fourier Transform :

$\widehat{\partial^\alpha f} \in L^1 \Rightarrow \partial^\alpha f$ is continuous

(Q: func^n in H^1 which is not continuous. (Friday HW To see how $s = \frac{n}{2}$ case is not cont)

Ans: $f(x,y) = \begin{cases} 1 & x>0 \\ 0 & \end{cases}$

~~Proof cont~~

No harmonic func^n in L^2 or C_c^∞ (Liouville)

If $\varphi \in L^2$

$-\Delta \varphi = \lambda \varphi$

$+|\xi|^2 \hat{\varphi} = \lambda \hat{\varphi} \qquad (\lambda + |\xi|^2)\hat{\varphi} = 0$

$\operatorname{supp} \hat{\varphi} \subseteq |\xi| = \sqrt{\lambda}$ → set of measure 0.

$\varphi = 0$ almost everywhere

<u>Proof Sketch</u>. $\Delta^\alpha f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\xi} \widehat{\Delta^\alpha f}(\xi)\, d\xi$

$\Delta^\alpha f(y) = \cdots$

$|\Delta^\alpha f(x) - \Delta^\alpha f(y)| = \frac{1}{(2\pi)^{n/2}} \left| \int_{\mathbb{R}^n} e^{ix\xi} (1 - e^{i(y-x)\xi}) (\widehat{\Delta^\alpha f}) d\xi \right|$

$\leq L_\infty(1 - e^{i(y-x)\xi}) (L_1 |\widehat{\Delta^\alpha f}|) \frac{1}{(2\pi)^{n/2}}$

as $x \to y$ → 0 ⇐ cont

To prove thm, enough to justify

$\|\Delta^\alpha f\|_{L^\infty} \underset{\text{upto const}}{\overset{F^{-1}: L^1 \to L^\infty}{\lesssim}} \|\widehat{\Delta^\alpha f}\|_{L^1} \leq \|f\|_{H^s} \quad \forall |\alpha| \leq k$

C^k norm

So $\|\widehat{\Delta^\alpha f}\|_{L^1} = \int_{\mathbb{R}^n} |\xi|^\alpha |\hat{f}(\xi)|\, d\xi$

$\leq \int_{\mathbb{R}^n} (1+|\xi|^2)^{k/2} |\hat{f}(\xi)|\, d\xi$

$$= \int_{\mathbb{R}^n} (1+|\xi|^2)^{s/2} |\hat{f}(\xi)| (1+|\xi|^2)^{\frac{k-s}{2}} d\xi$$

Holder
$$\leq \|f\|_{H^s} \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^{k-s} d\xi \right)$$

converges for $k-s > \frac{n}{2}$ (see S.. prey)

$$\left[\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 \right]^{1/2}$$

Corollary 2 Any $\omega \in \mathcal{E}'(\mathbb{R}^n)$ belongs to some $H^s(\mathbb{R}^n)$
(dist of compact supp)

Proof Let $\omega \in \mathcal{E}'(\mathbb{R}^n) = (C^\infty(\mathbb{R}^n))^*$ ← ω coming from dual
$\forall \varphi \in S(\mathbb{R}^n) \subseteq C^\infty(\mathbb{R}^n)$

$$|(\omega, \varphi)| \leq C_k \|\varphi\|_{C^k(B_R(0))}$$ (seminorm largest)

$$\leq C_k \|\varphi\|_{C^k(\mathbb{R}^n)}$$

Use Sobolev Embedding $\leq C_k' \|\varphi\|_{H^s}$, $s > \frac{n}{2} + k$

ω has been interpreted as a cont. linear funcnal on a dense subspace of $H^s(\mathbb{R}^n)$ and it extends (by density) to an element in $(H^s)^*$. so $\omega \in H^{-s}$
↑ (duality)

Another proof using dist of finite order.

$\delta_{x_1} + D(\delta_{x_2}) + D^2(\delta_{x_3}) + \cdots$ distributions of not finite order.
$x_1, x_2, \ldots x_n \to x_0$

→ can't extend dist. beyond this.

$S(\mathbb{R}^n)$ is dense in H^s

Trace Theorem

Attacking problem of pointwise interpreting distributions. If not pointwise can we have intersectⁿ when restricted to line, plane, boundary?

BVP H^{s} $\mathbb{R}^{n+1/2}$

Basically we are trying to approach the concept of "pointwise-defined" functions in a different way.

$$\mathbb{R}^n = \mathbb{R}^{n-k} \times \mathbb{R}^k$$
$$x \qquad y \qquad z$$

Fourier ξ $\qquad \eta \qquad \zeta$

restrictⁿ map $R: S(\mathbb{R}^n) \to S(\mathbb{R}^{n-k})$

$f \to Rf$

$Rf(y) = f(y,0)$

$R': S'(\mathbb{R}^{n-k}) \to S'(\mathbb{R}^n)$

THM If $s > \frac{k}{2}$, R extends to a bounded map $H^s(\mathbb{R}^n) \to H^{s-k/2}(\mathbb{R}^{n-k})$

"For every dimension reductions, you lose $\frac{1}{2}$ weak deriva"

March 29 Friday 3:30 pm

$$f \in H^s(\mathbb{R}^n)$$
$$Rf(y) = f(y,0)$$

$x = (y, z)$

$\mathbb{R}^n \quad \mathbb{R}^{n-k} \quad \mathbb{R}^k$

$\xi \quad \eta \quad \zeta$

$$R: H^s(\mathbb{R}^n) \longrightarrow H^{s-k/2}(\mathbb{R}^{n-k})$$

Also continuous

Proof: suffices to check

$$\|Rf\|_{H^{s-\frac{k}{2}}(\mathbb{R}^n)} \le C\|f\|_{H^s(\mathbb{R}^n)}$$

$$Rf(y) = f(y,z)|_{z=0} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{iy\cdot\eta} \hat{f}(\eta,\xi)\, d\eta\, d\zeta$$

$$\hat{{}^{R}f}(y) = f(y, z=\,)\Big|_{z=0} = \frac{1}{(2\pi)^{n/2}} \int_{R^n} e^{iy\cdot\eta} \hat{f}(\eta, \xi)\, d\eta\, d\xi$$

$$\shortparallel \qquad \downarrow \text{ Fubini}$$

$$\frac{1}{(2\pi)^{\frac{n-k}{2}}} \int_{R^{n-k}} e^{iy\cdot\eta}\, \hat{{}^{R}f}(\eta)\, d\eta = \frac{1}{(2\pi)^{n/2}} \int_{R^{n-k}} \left(\int_{R^k} (\ \ )\, d\xi \right) d\eta$$

$$\therefore \hat{{}^{R}f}(\eta) = \frac{1}{(2\pi)^{k/2}} \int_{R^k} \hat{f}(\eta, \xi)\, d\xi$$

$$\hat{{}^{R}f}(\eta) = \int_{R^k} \hat{f}(\eta, \xi)\,(1+|\eta|^2+|\xi|^2)^{s/2}\,(1+|\eta|^2+|\xi|^2)^{-s/2}\, d\xi$$

$$a^2 = 1+\eta^2$$

Holder

$$\hat{{}^{R}f}(\eta) \le \int_{R^k} \hat{f}(\eta, \xi)\,(1+|\eta|^2+|\xi|^2)^{s/2}\, d\xi$$

$$\int_{R^k} (1+|\eta|^2+|\xi|^2)^{-s}\, d\xi$$

$$\downarrow$$

$$\le (\qquad) \int_{R^k} (a^2+|\xi|^2)^{-s}\, d\xi$$

$$|S^{k-1}| \int_0^\infty (a^2+r^2)^{-s}\, r^{k-1}\, dr$$

$$= \int_{0}^{\infty} (a^2+r^2)^{-s}\, r^{k-1}\, dr$$

$$= \int_0^\infty \left(1+\frac{r^2}{a^2}\right)^{-s} a^{-2s}\, \frac{r^{k-1}}{a^{k-1}}\, a^{k-1}\, dr$$

$$= a^{k-2s} \int_0^\infty (1+t^2)^{s}\, t^{k-1}\, dt \qquad \frac{r}{a} = t, \quad dr = a\,dt$$

$$\int_0^\infty (1+t^2)^{-s}\, t^{k-1}\, dt$$

Converges if $s > \frac{k}{2}$

Finally $|\hat{R} f(\eta)|^2 \lesssim (1+|\eta|^2)^{\frac{k}{2}-s} \int \frac{|\hat{f}(\eta,\xi)|^2}{(1+|\eta|^2+|\xi|^2)^{-s}}$

April 2
Tue 2-3:30 pm

HW5 is up

Compact Sobolev Embedding

(Rellich-Kondrakov Lemma)

Def: $H_0^s(\Omega)$ | Ω: bounded domain in $\mathbb{R}^n$

$= \text{completion of } C_0^\infty(\Omega) \text{ in } H^s(\mathbb{R}^n)$

Remark: Later on, for boundary valued problems, if boundary conditions cannot be interpreted in a pointwise sense, then Dirichlet Boundary condition means membership in $H_0^s(\Omega)$

Compact Operator: X, Y are Banach Spaces

$T: X \to Y$ takes bdd sets in X to relatively compact sets in Y

$H_0^s(\Omega) \overset{i}{\hookrightarrow} H_0^t(\Omega)$ $s > t$.

i is a compact operator

Application: $E(u) = \int |\nabla u|^2 + |u|^2$

$0 \le E = \inf_{u \in H_0^1(\Omega)} E(u)$

Minimize $E(u)$ in $H_0^1(\Omega)$

$u_n \in H_0^1$ s.t. $E(u_n) \to E$, $\|u_n\|_{H_0^1}$ is bounded

u_n has a convergent subsequence $u_{n_k} \to u^* \in L_0^2$

ultimately we will prove ~~that~~ u^* attains $E(u^*) = E$

Proof Compact Sobolev Embedding

Let f_k be a bounded sequence in $H_0^s(\Omega)$, so

$$\|f_k\|_{H_0^s} \leq C < \infty$$

Trick: Choose $\varphi \in C_c^\infty(\mathbb{R}^n)$ s.t. $\varphi \equiv 1$ on Ω

so $\varphi f_k = f_k$

$$\hat{\varphi} * \hat{f}_k = \hat{f}_k$$

convol^n product^D (upto constant)

Claim 1 $\hat{f}_k$ has a subsequence converging uniformly on compact sets in $\mathbb{R}^n$.

Claim 2 result follows from claim 1.

Check 1. $\hat{\varphi} \in S(\mathbb{R}^n)$, $\hat{f}_k \in S^*(\mathbb{R}^n)$

$$\hat{\varphi} * \hat{f}_k \in C^\infty(\mathbb{R}^n)$$

2. $$\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)^s \leq 2^{|s|}(1+|\xi-\eta|^2)^{|s|}$$

$\forall \xi, \eta \in \mathbb{R}^n$, $s \in \mathbb{R}$

- Folland Lemma 6.10

We prove Claim 1.

Since $\hat{\varphi} * \hat{f}_k = \hat{f}_k$

$$\hat{f}_k(\xi) = \int \hat{\varphi}(\xi-\eta)\hat{f}_k(\eta)\,d\eta$$

$$(1+|\xi|^2)^{s/2}|\hat{f}_k(\xi)| \leq 2^{|s|/2}\int_{\mathbb{R}^n} |\hat{\varphi}(\xi-\eta)|\,(1+|\xi-\eta|^2)^{|s|/2}\,|\hat{f}_k(\eta)|\,(1+|\eta|^2)^{s/2}\,d\eta$$

Holder

$$\leq 2^{|s|/2}\|\varphi\|_{H^{|s|}(\mathbb{R}^n)}\,\|f_k\|_{H^s(\mathbb{R}^n)}$$

for $s<0$ flip variables $\hat{f}_k(\eta) = \int \hat{\varphi}(\eta-\xi)\hat{f}_k(\xi)\,d\xi$

$$(1+|\eta|^2)^{\frac{s}{2}}|\hat{f}_k(\eta)| \overset{\text{Ineq.}}{\leq} \int |\hat{\varphi}(\eta-\xi)| \ldots$$

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$\hat{f}_k(\xi)$ is equi bounded

Similarly $(1+|\xi|^2)^{s/2} |\partial_j \hat{f}|$

$$\leq 2^{|s|/2} \|x_j \varphi\|_{H^{|s|}(\mathbb{R}^n)} \|f_k\|_{H^s(\mathbb{R}^n)}$$

Since

$$\partial_j \hat{f}_k(\xi) = \widehat{x_j f_k} = \widehat{\varphi x_j f_k}$$

$$= \hat{\varphi}_{x_j} * \hat{f}_k \quad \text{same argument as before}$$

$$\downarrow \qquad \downarrow$$

$$\|\varphi_n\| \qquad \|f\|$$

This means $\hat{f}_k(\xi)$ is equi continuous

Arzela - Ascoli Thm f_k cont funcⁿs on $\mathbb{R}^n$

f_k equibounded + equi continuous

$\Downarrow$

f_{k_j} converges uniformly on compact sets

This establishes Claim 1

ie. we have found $\hat{f}_{k_j}$ converging uniformly on compact sets.

Claim 2 We want to prove f_{k_j} converges in H_0^t

For any $R>0$.

$$\|\hat{f}_{k_i} - \hat{f}_{k_j}\|_{H^t}$$

$$= \int_{|\xi|\leq R} |\hat{f}_{k_i} - \hat{f}_{k_j}|^2 (1+|\xi|^2)^t \, d\xi$$

$$+ \int_{|\xi|>R} |\hat{f}_{k_i} - \hat{f}_{k_j}|^2 (1+|\xi|^2)^t \, d\xi$$

$$\int_{|\xi|>R} |\hat{f}_{ki} - \hat{f}_{kj}|^2 (1+|\xi|^2)^{t-s} (1+|\xi|^2)^s \, d\xi \qquad \begin{array}{c} H_0^s \hookrightarrow H_0^t \\ t<s \end{array}$$

$$\leq (1+R^2)^{t-s} \int_{|\xi|>R} |\hat{f}_{ki} - \hat{f}_{kj}|^2 (1+|\xi|^2)^s \, d\xi$$

$$\leq \frac{1}{(1+R^2)^{s-t}} \left(\|f_{ki}\|^2_{H^s} + \|f_{kj}\|^2_{H^s} \right)$$

$$(\|f_k\|_{H^s} \leq C)$$

Def $H^k(\Omega)$ = completion of $C^\infty(\Omega)$ $\quad k \in \mathbb{N} \cup \{0\}$
w.r.t

$$\|f\|^2_{H^k(\Omega)} = \sum_{|\alpha| \leq k} \|\partial^\alpha f\|^2_{L^2}$$

Alternative def^n

$$W^k(\Omega) = \{ u \in L^2 \mid \partial^\alpha u \in L^2 \ \forall |\alpha| \leq k \}$$

Thm Serrin/Meyers

$$H^k(\Omega) = W^k(\Omega)$$

Define $\tilde{H}^k(\Omega)$ = completion of $C^\infty(\bar{\Omega})$ w.r.t $\|f\|_{H^k}$ -norm.

$$\tilde{H}^k(\Omega) \subseteq H^k(\Omega)$$

It turns out ~~that~~ the domain is not 'nice' then $\tilde{H}^k(\Omega) \neq H^k(\Omega)$. If "segment condition" is satisfied then $\tilde{H}^k = H^k$

Def. One way of defining $H^s(\Omega)$ for $s \notin \mathbb{N} \cup \{0\}$ is via "Complex Interpolation" (Taylor Ch. 4)

Def $H^s(\Omega) = H^s(\mathbb{R}^n) / H^s_{\overline{\mathbb{R}^n \setminus \Omega}}(\mathbb{R}^n)$ (Quotient out) all ~~that~~ are same inside are same.

where $H^s_K(\mathbb{R}^n) = \{ u \in H^s(\mathbb{R}^n) \mid \operatorname{supp} u \subseteq K \}$

Q. When $s \in \mathbb{N}$ do we get the original def^n.

Thm $(H_0^s(\Omega))^* = H^{-s}(\Omega)$ $s \geq 0$

Proof (Sketch) E is a Banach Space.
F is a closed subspace.

Then $F^* = E^*/F^\perp$ $F^\perp = \{u \in E^* \mid u(F) =$ dual

We want $F = H_0^s(\Omega)$ (is a Banach space)
is closed in $\underbrace{H^s(\mathbb{R}^n)}_{\longrightarrow E,\ \text{dual}}$

$$F^* = (H_0^s(\Omega))^* = H^{-s}(\mathbb{R}^n)/F^\perp$$

suffices to prove $F^\perp = H^{-s}_{\mathbb{R}^n \setminus \Omega}(\mathbb{R}^n)$

April 5
Fri
3:30p–5pm

Last Class: $(H^s(\Omega))^* = H^{-s}(\Omega)$

Elliptic Regularity

L^2-regularity for elliptic operators with C^∞ coefficients

Our operator will be $L = \sum_{|\alpha| \leq k} a_\alpha(x) \partial^\alpha$; $a_\alpha(x) \in C^\infty$

Def L is elliptic at x_0 if

$$\underbrace{\sum_{|\alpha|=k} a_\alpha(x_0)(i\xi)^\alpha}_{\text{principal symbol}} \neq 0 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}$$

Principal symbol

Eg: $\Delta = \partial x_1^2 + \ldots + \partial x_n^2$

principal symbol $-|\xi|^2$

Sublaplacian $\partial x_1^2 + \ldots + \partial x_{n-1}^2$ on $\mathbb{R}^n$

Principal symbol $-(\xi_1^2 + \ldots + \xi_{n-1}^2)$

$\overset{!!}{0}$ for $\xi_n \neq 0$ else all $\xi_i = 0$

Heat and Wave Eq^n are not elliptic (As in Sublaplacian)

Wave Eq^n: $\xi_1^2 - \xi_2^2 - \dots - \xi_n^2$

Remark Ellipticity depends only on the highest order terms

On a compact set $\Omega \subseteq \mathbb{R}^n$ if L is elliptic then one can assume:

$$\Big|\sum_{|\alpha|=k} a_\alpha(x)\xi^\alpha\Big| \geq A|\xi|^k$$

↳ independent of $x \in \Omega$

$$\frac{|\sum a_\alpha(x)\xi^\alpha|}{|\xi|^k} \in C(\Omega \times S_\xi^{n-1})$$

$\frac{\xi^\alpha}{|\xi|^k}$ $\to$ $|\alpha|=k$: $\xi \to \lambda\xi$ scale doesn't change.

Thm Suppose Ω is a bounded domain (open set) in $\mathbb{R}^n$ $L = \sum_{|\alpha|\leq k} a_\alpha(x)\partial^\alpha$ is elliptic on a nbd of $\bar{\Omega}$. Then for any $s \in \mathbb{R}$ $\exists\, C > 0$ s.t. $\forall u \in H_0^s(\Omega)$

$$\|u\|_{H^s} \leq C\left(\|Lu\|_{H^{s-k}} + \|u\|_{H^{s-1}}\right)$$

Remark This is the preliminary version of elliptic regularity

$L: H^s \longrightarrow H^{s-k}$ Elliptic Reg is opp

$\|Lu\|_{H^{s-k}} \leq C\|u\|_{H^s}$ (Proof: this holds for any diff. operator of order k)

Proof Three steps

- Prove for constant coefficient L, $a_\alpha(x)=0$ $|\alpha|<k$ (No lower order terms)
- Prove for variable coeff $L = \sum_{|\alpha|=k} a_\alpha(x)\partial^\alpha$ (difficult)
- Introduce lower order terms (done in Folland H.W)

1st step Suppose $a_\alpha = 0$ $\forall |\alpha| < k$

and a_α is constant for $|\alpha| = k$

For $u \in H^s$, $\widehat{Lu}(\xi) = (i)^k \sum_{|\alpha|=k} a_\alpha \xi^\alpha \hat{u}(\xi)$

$$|\widehat{Lu}(\xi)|^2 = |\sum_{|\alpha|=k} a_\alpha \xi^\alpha|^2 \, |\hat{u}(\xi)|^2$$

$$\geq A^2 |\xi|^{2k} |\hat{u}(\xi)|^2$$

$$|\widehat{Lu}(\xi)|^2 A^{-2} (1+|\xi|^2)^{s-k} \geq |\xi|^{2k} (1+|\xi|^2)^{s-k} |\hat{u}(\xi)|^2$$

Now $|\widehat{Lu}(\xi)|^2 A^{-2} (1+|\xi|^2)^{s-k} + (1+|\xi|^2)^{s-k} |\hat{u}(\xi)|^2$

$$\geq (|\xi|^{2k}+1)(1+|\xi|^2)^{s-k} |\hat{u}(\xi)|^2$$

$$\geq \frac{1}{2^k}(1+|\xi|^2)^k (1+|\xi|^2)^{s-k} |\hat{u}(\xi)|^2$$

Integrating on both sides and using $(1+|\xi|^2)^{s-k} \leq (1+|\xi|^2)^s$

$(1+x)^k \leq 2^k(1+x^k)$

AM GM $\quad \frac{1+x}{2} \leq ((1)^k + (x)^k)^{1/k}$

$x^r x^{k-r}$ 2^k terms $\quad$ take power k.

Step 2 Before proceeding to step 2 we need a few preliminary lemmas. ~~Note that~~ we can express the ~~number~~ norm of H^s in terms of

$$\Lambda^s = (1-\Delta)^{s/2} \quad ; \quad s \in \mathbb{R}$$

$$\widehat{\Lambda^s f}(\xi) = (1+|\xi|^2)^{s/2} \hat{f}(\xi)$$

Convince yourself: $\|f\|_{H^s} = \|\Lambda^s f\|_{L^2}$

$\|\Lambda^s f\|_{L^2} \underset{\text{Planch.}}{=} \|\widehat{\Lambda^s f}\|_{L^2} = \|(1+|\xi|^2)^{s/2}\hat{f}(\xi)\|_{L^2}$

$\stackrel{\text{def}}{=} \|f\|_{H^s}$

We want to compare $\partial^\alpha(\varphi u) \longleftrightarrow \varphi\partial^\alpha u$

const coeff. commutator variable coeff.

Now we derive a "commutator estimate"

<u>Lemma</u> $s \in \mathbb{R}$ $\sigma > \frac{n}{2}$ Then $\exists\, c = c_{s,\sigma}$

s.t. $\forall\ \varphi \in S(\mathbb{R}^n)$, $f \in H^{s-1}(\mathbb{R}^n)$;

$$\|[\Lambda^s, \varphi]f\|_{L^2} \le C\|\varphi\|_{H^{|s-1|+1+\sigma}}\|f\|_{H^{s-1}}$$

<u>Proof</u> set $f = \Lambda^{1-s}g$ $\quad(\Lambda^{s-1}f = g)$

To prove: $\|[\Lambda^s, \varphi]\Lambda^{1-s}g\|_{L^2} \le C\|\varphi\|_H\|g\|_{L^2}$

<u>Claim</u>: $(\Lambda^s\varphi\Lambda^{1-s}g - \varphi\Lambda g)^\wedge(\xi)$

$= \int K(\xi,\eta)\hat{g}(\eta)\,d\eta$ where

$$K(\xi,\eta) = \left[(1+|\xi|^2)^{s/2} - (1+|\eta|^2)^{s/2}\right]\hat{\varphi}(\xi-\eta)(1+|\eta|^2)^{\frac{1-s}{2}}$$

$$\widehat{\Lambda^s\varphi\Lambda^{1-s}g}(\xi) = (1+|\xi|^2)^{s/2}\,\widehat{\varphi\Lambda^{1-s}g}(\xi)$$

$$= (1+|\xi|^2)^{s/2}\,\hat{\varphi} * \widehat{\Lambda^{1-s}g}(\xi)$$

$= \ldots$

Real variables claim (MVT-type);

$$(1+|\xi|^2)^{s/2} - (1+|\eta|^2)^{s/2} \le |s|\,|\xi-\eta|\left[(1+|\xi|^2)^{\frac{s-1}{2}} + (1+|\eta|^2)^{\frac{s-1}{2}}\right]$$

$$K(\xi,\eta) \le |s|\,|\xi-\eta|\left((1+|\xi|^2)^{\frac{s-1}{2}}(1+|\eta|^2)^{\frac{1-s}{2}} + 1\right)|\hat{\varphi}(\xi-\eta)|$$

$$\underset{6.10}{\overset{\text{Lemma}}{\le}} |s|\,2^{\frac{|s-1|}{2}}\,|\xi-\eta|\,|\hat{\varphi}(\xi-\eta)|\left[1+(1+|\xi-\eta|^2)^{\frac{|s-1|}{2}}\right]$$

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Justify

$$\leq C_2 |\hat{\varphi}(\xi-\eta)| (1+|\xi-\eta|^2)^{\frac{|s-1|+1}{2}}$$

(Check: $a(1+a^2)^b \lesssim (1+a^2)^{b+\frac{1}{2}}$

Then, $\int |K(\xi,\eta)| d\xi$ and $\int |K(\xi,\eta)| d\eta$

are bounded above by Holder.

$$C_2 \left(\int |\xi \hat{\varphi}(\xi)|^2 \left(\int |\hat{\varphi}(\xi)|^2 (1+|\xi|^2)^{|s-1|+1+\sigma} d\xi \right)^{1/2} \right.$$

$$\left(\int (1+|\xi|^2)^{-\sigma} d\xi \right) \to \text{bound for } \sigma > \frac{n}{2}$$

$$= C_3 \|\varphi\|_{H^{|s-1|+1+\sigma}}$$

(*) <u>Theorem</u> L-p Boundedness of linear operators of the kind

$$Tf(x) = \int_X K(x,y) f(y) dy$$

$K(x,y)$ → measurable on $X \times X$

X — σ-finite measure space

If $\sup_{x\in X} \int_X K(x,y)\, dy \leq C$, $\sup_{y\in X} \int_X K(x,y) dx \leq C$

Then $T: L^p \to L^p$ $\|Tf\|_{L^p} \leq C \|f\|_{L^p}$

$1 \leq p \leq \infty$

<u>Lemma</u> $s \in \mathbb{R}$ $\sigma > \frac{n}{2}$ $\exists$ constant $C = C_{s,\sigma}$:

s.t. $\forall$ $\varphi \in S(\mathbb{R}^n)$ $f \in H^s(\mathbb{R}^n)$,

$$\|\varphi f\|_{H^s} \leq [\sup |\varphi(x)|] \|f\|_{H^s}$$

$$+ C \|\varphi\|_{H^{|s-1|+1+\sigma}} \|f\|_{H^{s-1}}$$

<u>Proof</u> $\|\varphi f\|_{H^s} = \|\Lambda^s \varphi f\|_{L^2} \leq \|\varphi \Lambda^s f\|_{L^2} + \|[\Lambda^s, \varphi] f\|_{L^2}$

$\leq \|\varphi\|_{L^\infty} \|\Lambda^s f\|_{L^2} + \ldots$

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$L = \sum_{|\alpha|=k} a_\alpha(x)\partial^\alpha$

$u \to \varphi u$ — Bump function supported compactly around x_0

$L_{x_0} = \sum_{|\alpha|=k} a_\alpha(x_0)\partial^\alpha$

$\|Lu - L_{x_0}u\|_{H^{s-k}} \leq \|L_{x_0}u\|$

Partition of Unity : stitch together local smoothness to get Global smoothness

1. 3 classes Next Week Mon 11-1 5:30 - 8
2. PPT Friday : Vaibhav, Saumyajit Sat: The Rest.
3. Final Exam (2nd May?)
4. Final Miscellaneous HW after Exam. 4th May.

April 9
Tue 3:30-5pm

Continue with Step 2

Assume $a_\alpha = 0$ for $|\alpha| < k$; $a_\alpha \in C^\infty$, $|\alpha| = k$

$\forall x_0 \in \Omega$ define $L_{x_0} = \sum_{|\alpha|=k} a_\alpha(x_0)\partial^\alpha$

since $\bar{\Omega}$ is compact, $A = \min_{x\in\Omega} \dfrac{|\sum a_\alpha(x)\xi^\alpha|}{|\xi|^k}$

is independent of $x \in \Omega$

Freezing $x = x_0$ makes it constant coeff.

from Step 1 $\rightarrow$ $\|u\|_{H^s} \leq C_0\left(\|L_{x_0}u\|_{H^{s-k}} + \|u\|_{H^{s-1}}\right)$

Note: C_0 is independent of x_0

Assume first that $u \in H_0^s(\Omega)$ is supported in a small nbd of x_0

Want to estimate $\|Lu - L_{x_0}u\|_{H^{s-k}}$

WLOG $a_\alpha \in C_c^\infty(\mathbb{R}^n)$

So $|a_\alpha(x) - a_\alpha(x_0)| \leq C_1|x - x_0|$

C_1 independent of x

(Lipschitz)

set $\delta = \dfrac{1}{4n^k C_0 C_1}$ and fix $\varphi \in C_c^\infty(B_{2\delta}(0))$

with $0 \leq \varphi \leq 1$ and

$\varphi \equiv 1$ on $B_\delta(0)$

Suppose $\operatorname{supp} u \subseteq B_\delta(x_0)$

For any $\alpha = (\alpha_1, \alpha_2, \dots \alpha_n)$ $\qquad |\alpha| = k$

$$(a_\alpha(x) - a_\alpha(x_0))\partial^\alpha u(x)$$

$$= \underbrace{\varphi(x - x_0)(a_\alpha(x) - a_\alpha(x_0))}_{\psi_{x_0 \alpha}(x)}\partial^\alpha u(x)$$

Then $\psi_{x_0,\alpha}(x) \le C_1(2\delta) = (2n^k C_0)^{-1}$

By the previous Lemma:

$$\| a_\alpha(x) - a_\alpha(x_0)\partial^\alpha u\|_{H^{s-k}} = \| \psi_{x_0,\alpha}\partial^\alpha u \|_{H^{s-k}}$$

$$\underset{\text{Lemma}}{\le} \|\psi_{x_0\alpha}\|_{L^\infty} \|\partial^\alpha u\|_{H^{s-k}} + C_2\|\partial^\alpha u\|_{H^{s-k-1}}$$

H^s norm stays same even if we shift

$$\le (2n^k C_0)^{-1} \|u\|_{H^s} + C_2\|u\|_{H^{s-1}}$$

$\partial^\alpha . H^s \to H^{s-k}$ bdd

$\nabla \quad H^1 \to L^2 \longrightarrow$ all derives ~~upto~~ order H^{s-k}. [illegible]

$\|\nabla u\|_{L^2} \le \|u\|_{H^1}$

Digression $\qquad \|\psi_{x_0\alpha}\|_{H^s} \le C$

choose m just above s (to get integer)

$$\|\psi_{x_0,\alpha}\|_{H^m} \le C$$

$\|\nabla^\alpha \psi_{x_0}\|_{L^2}$ $\qquad \int$ change of var. $x - x_0$

Then dominate H^m by H^s $\qquad L^2_{x-x_0}$

Only for non-integer s: F.T.

So $\|Lu - L_{x_0} u\|_{H^{s-k}} \le \sum_{|\alpha|=k} \|a_\alpha(x) - a_\alpha(x_0)\partial^\alpha u\|_{H^{s-k}}$

$\le \varepsilon C_0^{-1} \|u\|_{H^s} + n^k C_2 \|u\|_{H^{s-1}}$ (ii)

(n^k terms)

Using i) and (ii) and choosing C_0 large

$$\|u\|_{H^s} \le C_3 \left(\|Lu\|_{H^{s-k}} + \|u\|_{H^{s-1}}\right) \quad \ldots iii)$$

Recall iii) holds for u supported in $B_\delta(x_0)$

Now we bring in a partition of unity argument.

Take an open cover of Ω

$$\Omega \subseteq \bigcup_{i=1}^{N} B_\delta(x_i)$$

Paracompact spaces $\Leftrightarrow$ POU. Stones: Any metric space is paracompact

and then take a partition of unity subordinate to this open cover.

$\exists$ smooth functions $\varphi_i \in C_c^\infty(B_\delta(x_i))$

s.t. $0 \le \varphi_i \le 1$ and $\sum_{i=1}^{N} \varphi_i = 1$

Remark: Philosophically partition of Unity is used to stitch local smooth data into global smooth data.

For $u \in H_0^s(\Omega)$ $\varphi_j u \in H_0^s(\Omega)$

$\varphi_j u \in H_0^s(\Omega)$ and $\operatorname{supp} \varphi_j u \subseteq B_\delta(x_j)$

Now $\|u\|_{H^s} = \|\sum_{i=1}^{N} \varphi_i u\|_{H^s}$ $(\because \sum_i \varphi_i = 1)$

$$\le \sum_{i=1}^{N} \|\varphi_i u\|_{H^s}$$

Claim: $\|[L, \varphi] u\|_{H^{s-k}} \le C_4 \|u\|_{H^{s-1}}$

$[L, \varphi]$ D.Op. of order $k-1$ $\rightarrow H^{s-k+(k-1)} = H^{s-1}$

$L = \sum_{|\alpha|=k} a_\alpha \partial^\alpha$ $\quad L(\varphi u) = \sum_{|\alpha|=k} a_\alpha \partial^\alpha (\varphi u) = \sum a_\alpha \varphi \partial^\alpha u$ + L.o terms in deriv of u

$\varphi L u = \sum_{|\alpha|=k} a_\alpha \varphi \partial^\alpha u$ $\quad (L\varphi - \varphi L) u =$ l.o.t in

Now iii)

$$\leq C_3 \sum_{i} \left(\|L \varphi_i u\|_{H^{s-k}} + \|\varphi_i u\|_{H^{s-1}} \right)$$

$$\leq C_3 \sum \left(\|\varphi_i L u\|_{H^{s-k}} + \|[L, \varphi_i] u\|_{H^{s-k}} + \|\varphi_i u\|_{H^{s-1}} \right)$$

φ is C^∞ we can just absorb it in C_3

Lemma: Ω open + Usual set up $u \in H^s_{loc}(\Omega)$ and $Lu \in H^{s-k+1}_{loc}(\Omega)$ ~~then~~ $u \in H^{s+1}_{loc}(\Omega)$

Proof ~~Def~~ $u \in H^s_{loc}$ iff $\varphi u \in H^s(\mathbb{R}^n)$ $\forall$ $\varphi \in C_c^\infty(\Omega)$

Proof It suffices to prove ~~that~~ $\psi u \in H^{s+1}$ $\forall$ $\psi \in C_c^\infty(\mathbb{R}^n)$

By hypothesis, $\psi u \in H^s$

and $\psi L u \in H^{s-k+1}$

clearly $[L, \psi] u \in H^{s-k+1}$ $[L, \psi]$ of order $k-1$ (~~lower of the two~~) $\therefore H^{s-k+1}$

We want to prove ~~$Lu \in L^2_{loc}$~~ $\forall u \in H^s$ then $u \in H^{s+1}$

$$L(\psi u) = \psi L u + [L, \psi] u \in H^{s-k+1}$$

Finite difference

Define $\Delta_h^j f = \dfrac{f(x+he_j) - f(x)}{h}$

If $f \in \mathcal{D}'(\Omega)$ Action of distrib^n is given by $(f(x+he_j), \varphi) = (f, \varphi(x-he_j))$

Lemma: $L = \sum_{|\alpha| \leq k} a_\alpha \partial^\alpha$, $a_\alpha \in S(\mathbb{R}^n)$

$\forall s \in \mathbb{R}$, $[\Delta_h^\beta, L] : H^s \to H^{s-k-|\beta|+1}$

(Folland 6.20)

Lemma Y $\|\partial^\alpha f\|_{H^s} = \lim_{|h|\to 0} \sup \|\Delta^\alpha_h f\|_{H^s}$

$$\|\Delta^j_h(\Psi u)\|_{H^s} \underset{\substack{\text{last}\\\text{thm}}}{\leq} c\left(\|L\Delta^j_h(\Psi u)\|_{H^{s-k}} + c\|\Delta^j_h(\Psi u)\|_{H^{s-1}}\right)$$

Lemma X

$$\leq C\left(\|\Delta^j_h L(\Psi u)\|_{H^{s-k}} + C\|\Psi u\|_{H^s} + \|\Delta^j_h(\Psi u)\|_{H^{s-1}}\right)$$

Lemma Y

$$\leq \|\nabla(\Psi u)\|_{H^s} < \infty$$

$$\to \Psi u \in H^{s+1}$$

Presentation Friday

April 15 Monday 11am 1pm

Elliptic Regularity Theorem

Usual setup. (not need bounded $\because$ loc)

$Lu = f$ where $f \in H^s_{loc}(\Omega)$

then $u \in H^{s+k}_{loc}(\Omega)$ $\quad$ k: degree of oper.

Corollary $\Delta u = 0$

or $Lu = 0$ $\quad$ L: Elliptic

$\Downarrow$

$u \in H^s_{loc}\ \forall s$ $\quad$ ($\because 0 \in H^s_{loc}$ forall s)

$\Downarrow$ sobolev Embedding $\to C^k$

u is smooth

Corollary 2. Elliptic operators with smooth coefficients are hypoelliptic.

(smooth coeff $Lu \in H^s_{loc} \to u \in H^s_{loc}$ Hypoelliptic.)

Proof Let $\varphi \in C_c^\infty(\Omega)$

T.P. $\varphi u \in H^{s+k}$

Choose $\psi_0 = \psi \in C_c^\infty(\Omega)$

s.t. $\psi \equiv 1$ on supp φ

1st Observation:

$$\underset{\substack{C_c^\infty \\ \text{dist}}}{\psi u} \in \mathcal{E}'(\mathbb{R}^n) \longrightarrow \text{dist. of compact support}$$

$$\Downarrow$$

$\psi u \in H^t$ for some t (proved in class)

By decreasing t if necessary, assume:

$$N = s+k-t \in \mathbb{N}$$

Proceeding inductively, choose C^∞ functions

$\psi_1, \psi_2, \dots, \psi_{N+1}$ s.t.

$$\text{supp } \psi_j \subseteq \{\psi_{j-1} = 1\}$$

and $\psi_j = 1$ on a nbd of supp φ.

We want to prove by induction that $\psi_j u \in H^{t+j}$.

Initial case $\psi_0 u \in H^t$

suppose it is ~~true~~ for some j. i.e.

$\psi_j u \in H^{t+j}$ for some j $0 \le j \le N$

$$\boxed{\psi_{j+1}\psi_j u = \begin{cases} \psi_{j+1} u & \in H^{t+j} \\ 0 & \end{cases}}$$

$$L(\psi_{j+1} u) = \underset{\substack{\Uparrow \\ H^s}}{\psi_{j+1} Lu} + [L, \psi_{j+1}]u$$

$$[L, \psi_{j+1}]u = [L, \psi_{j+1}](\psi_j u) \in H^{t+j-k+1}$$

Then $\boxed{L(\psi_{j+1} u) \in H^{t+j-k+1}}$

$\because H^s \subseteq H^{t+j-k+1}$ $\quad N = s+k-t$

$\Updownarrow$

$s \ge t+j-k+1 \to N \ge j+1$ remember $j \le N-1$

By previous lemma $\varphi_{j+1} u \in H^{t+j+1}$

Note: If the coeff of L are real analytic and f is real analytic

Real Analytic locally extend to complex analytic

$f(x) = \sum a_n x^n \longrightarrow f(z) = \sum a_n z^n$

No part^n of unity to go global.

vanishes at most points non-zero at some x in real analytic

Existence / Uniqueness and Eigenfunctions

suppose $\Omega \subseteq \mathbb{R}^n$ is bounded domain with smooth boundary. Let $L = \sum_{|\alpha| \le 2m} a^\alpha \partial^\alpha$ be a strongly elliptic operator.

Def L strongly elliptic operator on $\bar{\Omega}$

if $(-1)^m \operatorname{Re} \sum_{|\alpha|=2m} a_\alpha(x) \xi^\alpha \geq C |\xi|^{2m}$

$\forall \xi \in \mathbb{R}^n, x \in \bar{\Omega}$

Remark 1: Observe that $-\Delta$ is strongly elliptic, but Δ is not.

2. *Thm Hörmander (in Folland)

Any scalar elliptic operator is even order

Cauchy Riemann is elliptic system.

Dirac operator.

3. Folland discussion why defn of strongly ellipticity is somewhat natural, one can look at Ch 7. of Folland which shows problems coming up with many other "natural" candidates.

Def: The formal adjoint of L is the differential operator L^* on Ω satisfying $(L^* u, v) = (u, Lv)$ $\forall (u, v) \in C_c^\infty(\Omega)$ where $(u, v) = \int_\Omega u \bar{v}$

Note: 1. Check by Integration by parts that

$$L^* v = \sum_{|\alpha| \le 2m} (-1)^{|\alpha|} \partial^\alpha (\bar{a}_\alpha v)$$

2. check that L strongly elliptic $\Leftrightarrow$ L^* strongly ellip.

Def: Dirichlet Form

(IBP every out. order)

A sesquilinear form of the type:

$$D(u, v) = \sum_{|\alpha|, |\beta| \le m} (\partial^\alpha u, a_{\alpha\beta} \partial^\beta v)$$

sesquilinear: linear in 1st, conjugate linear in 2nd $\quad a_{\alpha\beta} \in C_c^\infty(\bar{\Omega})$

is called a $\boxed{\text{Dirichlet form}}$ $(u, v \in C_c^\infty(\Omega))$ form of order m.

D is said to be the Dirichlet form for operator L if $D(u, v) = (Lu, v) \quad \forall u, v \in C_c^\infty(\Omega)$

Ex: $L = -\Delta$

$$D(u, v) = \sum_j (\partial_j u, \partial_j v) \longleftarrow (-\Delta u, v) = \int \nabla u \nabla v$$

Remark: 1. Dirichlet form for an operator L may not be unique

$$D'(u, v) = ((\partial_x + i\partial_y)u, (\partial_x + i\partial_y)v)$$

is another DF for $L = -\Delta$

2. $L = \Delta^2$ on $\mathbb{R}^2$

$$D_1 = (\Delta u, \Delta v) \qquad D_2 = ((\partial_x^2 - \partial_y^2)u, (\partial_x^2 - \partial_y^2)v)$$

Imp Definition: A dirichlet form DF is said to be strongly elliptic on $\bar{\Omega}$ if $\exists c > 0$

s.t. $\quad \mathrm{Re} \sum_{|\alpha| = |\beta| = m} a_{\alpha\beta}(x) \xi^{\alpha+\beta} \ge c|\xi|^{2m} \quad \forall \xi \in \mathbb{R}^n \; \forall x$

Check: L strongly elliptic $\Leftrightarrow$ every DF for L is strongly elliptic.

Let us now reformulate Dirichlet Problem.

Given $f \in L^2(\Omega)$, find $u \in H_0^m(\Omega)$ s.t.

$$D(v, u) = (v, f) \quad \forall v \in H_0^m(\Omega)$$

Remark: 1. $H_0^m \to$ Dirichlet boundary condition

generalize: $\underset{\text{order } 2m}{L}u = f$ B.C. $B_j u = g_j$ on $\partial\Omega$ $\hookrightarrow$ order $2m-1$ (Taylor ch5.11)

2. We already know from elliptic regularity that $u \in H^{2m}_{loc}(\Omega)$ provided $Lu = f$ can be solved $\longrightarrow f \in L^2$ then $u \in H^{2m}_{loc}(\Omega)$

But now our reformulation, we look for $u \in H_0^m(\Omega)$.

Now u, v both in $H_0^m(\Omega)$ set up Hilbert space and Reisz Rep. Thm will give what we need.

Def (Coercivity). A Dirichlet form D of order m on $\overline{\Omega}$ is said to be coercive over $H_0^m(\Omega)$ if $\exists$ constants $C > 0$, $\lambda \geq 0$ s.t. $\operatorname{Re} D(u, u) \geq C\|u\|^2_{H_0^m(\Omega)} - \lambda\|u\|^2_{L^2}$ $\forall u \in H_0^m(\Omega)$

Almost +ve def.

Elliptic operators are elliptic because of highest order terms. No control on lower order terms.

$D - 1 \longrightarrow \geq C\|u\|^2 - 1\|u\|^2_{L^2}$

$D'(u, u) = D(u, u) + \lambda(u, v)$ shift spectrum + becomes +ve def.

Relation of strongly elliptic with coercivity is given by Garding's Inequality.

Thm Garding's Inequality

Let D be a strongly elliptic Dirichlet form of order m on $\overline{\Omega}$ $\Longrightarrow$ D is coercive over $H_0^m(\Omega)$

Converse is also true but not needed for us.

$\hookrightarrow$ Proof in Folland.

Proof of Garding's Inequality

Sketch of Proof: similar to Elliptic Regularity

Step 1: Prove statement for $\Delta(u,u) = \sum_{|\alpha|=m, |\beta|} (a_{\alpha\beta} \partial^\alpha ...)$

Top order terms where $a_{\alpha\beta}$ are const. (Folland)

Step 2: Observe that it suffices to establish that

for $u \in C_c^\infty(\Omega)$ $\quad \mathrm{Re}\,\Delta(u,u) \geq c\|u\|_{H_0^m} - \lambda\|u\|_{L^2} - c_2\|u\|_{H^m}\|u\|_{L^2}$

By density, just check for C_c^∞

Tacit claim: $\quad \|u\|_{H^m}\|u\|_{H^{m-1}} \leq c_1\|u\|_{H^m}^2 - c_3\|u\|_{L^2}^2$

$$\leq \underset{\text{Cauchy Schwartz}}{\frac{\varepsilon}{2}}\|u\|_{H^m}^2 + \frac{1}{2\varepsilon}\|u\|_{H^{m-1}}^2$$

Poincare

Recap: Dirichlet form: integrate L m times by parts

strong ellipticity

Bridge = Garding's Ineq.

coercivity $\rightarrow$ Reis ξ Rep $\leftarrow$

4/15 Mon. 5:30 – 8:30pm

Recall:

Δ: $\rightarrow$ Dirichlet form for a strongly elliptic operator L of order $2m$. $\quad \partial_\alpha \mapsto \xi_\alpha$

$\underset{f(i)}{\Delta\alpha} \rightsquigarrow \xi_\alpha$ get rid of $(-1)^m$

Reformulation: Given $f \in L^2(\Omega)$, find $u \in H_0^m(\Omega)$ s.t. $(v, f)_{L^2} = \Delta(v,u) \quad \forall v \in H_0^m(\Omega)$

Defined: strongly elliptic ~~operators~~ DF

associated to ~~simply~~ strongly elliptic operators.

Coercive DF: Δ is coercive over $H_0^m(\Omega)$

$$\Delta(u,u) \geq c_1\|u\|_{H_0^m(\Omega)}^2 - \lambda\|u\|_{L^2}^2 \quad \forall u \in H_0^m(\Omega)$$

Garding's Inequality: A strongly elliptic $\Updownarrow$ A coercive (converse true but not needed)

Proof step 1 Check for
$$A(u,v) = \Big(\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\, \partial^\alpha u, \partial^\beta v\Big) \quad a_{\alpha\beta} \text{ const.}$$

Step 2

Claim It is enough to establish:
$$\operatorname{Re} A(u,u) \geq C_2 \|u\|^2_{H_0^m} - \mu \|u\|^2_{L^2} - C_3 \|u\|_{H^m} \|u\|_{H^{m-1}}$$

$$\Big(\langle Lu, v\rangle = \int \sum_{|\alpha|\leq 2m} a_\alpha(x)\, \partial^\alpha u\, \bar{v}\, dx$$

By integr^ By Parts:
$$(Lu,v) = \int \sum_{|\alpha|,|\beta|\leq m} a_{\alpha\beta}\, \partial^\alpha u, \overline{\partial^\beta v}$$

Now
$$(Lu,u) = \sum_{|\alpha|=|\beta|=m} (a_{\alpha\beta}\, \partial^\alpha u, \partial^\beta v) + \sum_{\min\{|\alpha|,|\beta|\}<m} (a_{\alpha\beta}\, \partial^\alpha u\, \partial^\beta u)$$

Claim: One can ignore second ~~of~~ term:
$$\Big|\int a_{\alpha\beta}\, \partial^\alpha u\, \partial^\beta u\Big| \overset{\text{Hölder}}{\leq} \tilde{C} \|u\|_{H^{|\alpha|}} \|u\|_{H^{|\beta|}}$$
$$\leq \tilde{C} \|u\|_{H^m} \|u\|_{H^{m-1}}$$
$\hookrightarrow$ absorb

Focus on the 1st term:
$$\sum_{|\alpha|=|\beta|=m} (a_{\alpha\beta}\, \partial^\alpha u, \partial^\beta u)$$

Take a finite open cover of Ω and a finite partition of unity $\{\zeta_j\}$ subordinate to this open cover. ($\because \Omega$ is bounded)

For any f: $\int f = \int \sum_j \zeta_j f$

$\int \zeta_j a_{\alpha\beta} \partial^\alpha u \partial^\beta \overline{u}$

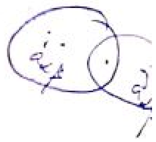
$= \zeta_j (a_{\alpha\beta} - a^j_{\alpha\beta}) \partial^\alpha u \partial^\beta \overline{u}$

$+ \int \zeta_j a^j_{\alpha\beta} \partial^\alpha u \partial^\beta \overline{u}$

where "frozen coeff" $a^j_{\alpha\beta}$ are defined as

$a^j_{\alpha\beta} = a_{\alpha\beta}(x_j)$ $\quad x_j \in \operatorname{supp} \zeta_j$

Assume that the cover is fine enough s.t.



$\zeta_j |a_{\alpha\beta} - a^j_{\alpha\beta}| \le \varepsilon \quad \forall j$

so the integral $\int \zeta_j (a_{\alpha\beta} - a^j_{\alpha\beta}) \partial^\alpha u \partial^\beta \overline{u}$

can be ignored

$\int | \zeta_j (a_{\alpha\beta} - a^j_{\alpha\beta}) \partial^\alpha u \partial^\beta \overline{u} | \le \varepsilon \int |\partial^\alpha u \partial^\beta u| \le \varepsilon \|u\|^2_{H^m}$

choose small enough ε.

Now we have to tackle $\int \zeta_j a^j_{\alpha\beta} \partial^\alpha u \partial^\beta \overline{u}$

Take:

$\eta_j^2 = \zeta_j$

$= \int a^j_{\alpha\beta} \eta_j^2 \partial^\alpha u \partial^\beta \overline{u}$

$= \int a^j_{\alpha\beta} \eta_j \partial^\alpha u \, \eta_j \partial^\beta \overline{u}$

$= \int a^j_{\alpha\beta} ([\eta_j, \partial^\alpha] u) \eta_j \partial^\beta \overline{u}$

$+ \int a^j_{\alpha\beta} \partial^\alpha (\eta_j u) \eta_j \partial^\beta \overline{u}$

$\cancel{+ \int a_{\alpha\beta}}$

$$= \int a^j_{\alpha\beta} \left([\eta_j, \partial^\alpha] u \right) \eta_j \, \partial^\beta \bar{u} + \int a_{\alpha\beta} \partial^\alpha (\eta_j u) [\eta_j, \partial^\beta] \bar{u}$$

$$+ \int a_{\alpha\beta} \, \partial^\alpha (\eta_j u) \partial^\beta (\eta_j u)$$

Observe that

$$\int |\partial^\beta u|^2 \leq \|u\|^2_{H^m}, \quad \int |[\eta_j, \partial^\alpha] u|^2$$

so we can ignore the first two $\leq \|u\|^2_{H^{m-1}}$ terms above.

Suffices to control

$$\int a^j_{\alpha\beta} \partial^\alpha (\eta_j u) \partial^\beta (\eta_j u)$$

Using step 1:

$$\operatorname{Re} \int a^j_{\alpha\beta} \, \partial^\alpha (\eta_j u) \partial^\beta (\eta_j u)$$

$$\geq c_1 \|\eta_j u\|^2_{H^m} - \lambda \|\eta_j u\|^2_{L^2}$$

The only thing left to check:

$$c_1 \sum \|\eta_j u\|^2_{H^m} - \lambda \sum \|\eta_j u\|^2_{L^2}$$

$$\geq c_1 \|u\|^2_{H^m} - \lambda \|u\|^2_{L^2} \qquad = \quad \sum_j \eta_j^2 = 1$$

Contains terms. when derivative doesn't hit η_j and more terms

Lax Milgram Lemma

Suppose H is a Hilbert space and $\Delta : H \times H \to \mathbb{C}$ is a sesquilinear form. Suppose $\exists$ constants $c_1, c_2 > 0$

s.t. $|\Delta(u,v)| \leq c_1 \|u\| \|v\|$ (Cauchy Schwartz)

$$|\Delta(u,u)| \geq c_2 \|u\|^2$$

Then $\exists$ invertible bounded operators

$\phi : H \to H$ and $\psi : H \to H$ s.t.

$$\langle v, w \rangle_{H_0} = \Delta(v, \phi w) = \overline{\Delta(\psi(w), v)}$$

Proof Fix $u \in H$, look at the map $H \to \mathbb{C}$

$$v \longmapsto \Delta(v,u)$$

This map is bounded linear funct^l

$$|\Delta(u,v)| \leq c_1 \underbrace{\|u\|}_{\text{fixed}} \|v\|$$

By Reisz Representation $\exists!$ Ru s.t.

$$\Delta(v,u) = \langle v, Ru \rangle \qquad \forall v \in H.$$

Claim $R \quad H \to H$ is ~~invertible~~ bounded.

$$\|Ru\|^2 = \langle Ru, Ru \rangle = \Delta(Ru, u)$$

$$\leq c_1 \|Ru\| \|u\|$$

$$\|Ru\| \leq c_1 \|u\|$$

Claim: R is injective

$$\|Ru\| \|u\| \overset{C.S.}{\geq} |\langle u, Ru \rangle|$$

$$= |\Delta(u,u)|$$

$$\geq c_2 \|u\|^2$$

$$\|Ru\| \geq c_2 \|u\| \longrightarrow$$ Nothing non zero can go to zero: Injective

Claim: R is surjective

Take $w \perp$ Range R

$\langle w, Ru \rangle = 0 \quad \forall u \in H$

$\Downarrow$

$|\langle w, Ru \rangle| = 0 \quad \forall u \in H$

$\Downarrow$

$|\Delta(w, u)| = 0 \quad \forall u \in H$

Use in particular $u = w$

$c_2 \|w\|^2 \leq |\Delta(w, w)| = 0$

Take $\phi = R^{-1}$.

Existence and Uniqueness Thm

Δ is an order m DF strictly coercive over $H_0^m(\Omega)$

$Re\, \Delta(u, u) \geq c_1 \|u\|^2_{H^m}$

There is a bounded injective operator

$A : L^2(\Omega) \longrightarrow H_0^m(\Omega)$

s.t. $\Delta(v, Af) = (v, f)_{L^2} \quad \forall v \in H_0^m(\Omega)$, $f \in L^2(\Omega)$

$\quad \parallel$

(v, Lu)

Remark Want to solve:

$(v, Lu) = (v, f)$

$\quad \parallel$

$\Delta(v, u) \qquad u = Af$ solves the Dirichlet problem.

Proof $H = H_0^m(\Omega)$

Δ satisfies Lax-Milgram cond^n:

$|\Delta(v, u)| = |\int a_{\alpha\beta} \partial^\alpha v \, \partial^\beta u \, \xi| \to$ Bounded smooth

$\leq c \|u\|_{H^m} \|v\|_{H^m}$

$|\Delta(u, u)| \underset{\text{coercivity}}{\geq} c_2 \|u\|^2_{H^m}$

$\exists \phi$ bounded s.t.

$\forall v, w \in H_0^m(\Omega)$ Lax milgram

$\langle v, w \rangle_{H_0^m(\Omega)} = \Delta(v, \phi w)$

We want L^2 norm.

Now $H_0^m(\Omega) \to \mathbb{C}$ $\quad v \mapsto (v, f)_{L^2}$

is a bdd linear functional

$|(v,f)_{L^2}|_{\mathbb{R}} \le \|v\|_{L^2} \|f\|_{L^2} \le \| v \|_{H^m} \|f\|_{H_0}$

$\exists!\ Rf \in H_0^m(\Omega)$ s.t.

$\cancel{\alpha}(v, f)_{L^2} = (v, Rf)_{H_0^m}$ | Riesz

Take $A = \phi_0 R$

$\Delta(v, Af) \underset{?}{=} (v, f)_{L^2}$

$\Delta(v, \phi_0 Rf) = (v, Rf)_{H_0^m(\Omega)}$

Remark: Proved existence uniqueness for strictly coercive Dirichlet form DF. However Elliptic operators in general $\rightsquigarrow$ coercive DF

$\operatorname{Re} \Delta'(u, v) \ge \|u\|^2_{H_0^m} - \lambda \|u\|^2_{L^2}$

$\Delta(u, v) := \Delta'(u, v) + \lambda(u, v)$ $\quad$ ↳ H^m bigger than H_0^m but norms are equal.

(Δ is strictly coercive covered by last result Folland)

April 16 Tuesday 3:30–5pm

Thm: Ω bounded. Suppose Δ is coercive over $H_0^m(\Omega)$ and $\Delta = \Delta^*$

$\Delta^*(u, v) := \overline{\Delta(v, u)}$

check that Δ^* is a Dirichlet ~~form~~ for L^* $\quad$ $f(x,y) = f(y,x)$

Then $\exists$ a complete orthonormal basis $\{u_j\}$ of $L^2(\Omega)$ consisting of eigenfunctions ie. $\forall j$ $u_j \in H_0^m(\Omega)$ and $\mu_j \in \mathbb{R}$ st

i) $\Delta(v, u_j) = \mu_j (v, u_j)$ $\quad \forall v \in H_0^m(\Omega)$

$\quad \| L(v, Lu_j)$

ii) $\mu > -\lambda \to \operatorname{Re}(\Delta(u, u)) \ge c \|u\|^2_{H^m} - \lambda \|u\|^2_{L^2}$

iii) $\lim_{j\to\infty} \mu_j = \infty$ $\qquad u_j \in C^\infty(\Omega)$

Proof Define $B'(v,u) := B(v,u) + \lambda(v,u)$

Let A and B be the solution operators for B' and $(B')^*$

Let $T = i\circ A \qquad S = i\circ B$

$i: H_0^m(\Omega) \to L^2(\Omega)$ is the inclusion operator.

Claim: By Rellich–Kondrachov / Compact Sobolev Embedding S and T are compact. $\qquad A, B : L^2 \to H_0^m$

(on $\mathbb{R}^n$ counterexample: $H^1(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$ find sequence with no convergent subsequence)

Claim $S = T^*$ $\quad \forall f, g \in L^2(\Omega)$

$$B(Bg, Af) = (Bg, f)_{L^2} = (Sg, f)_{L^2}$$

$$\|$$

$$B^*(Bg, Af) = (g, Af)_{L^2} = (g, Tf)_{L^2}$$

$\Rightarrow S = T^*$

So T is compact and self-adjoint?

(By Spectral theorem:
$\exists$ complete orthonormal basis $u_j \in L^2(\Omega)$
and $\alpha_j \geq 0 \quad Tu_j = \alpha_j u_j$

$\rightarrow$ we want $A = B \rightarrow S = T$

$B' = (B')^*$
$\uparrow$
$B = B^*$

$(B')^*(u,v) := \overline{B'(v,u)}$
$= \overline{B(v,u) + \lambda(v,u)}$
self adj. $= \overline{B^*(v,u) + \lambda(v,u)}$
$= \overline{B^*(v,u)} + \overline{\lambda(v,u)}$
$= B(u,v) + \overline{\lambda(v,u)}$
$= B(u,v) + \lambda(u,v)$
$= B'(u,v)$

$\therefore S = \boxed{T = T^*}$
self adjoint

By Spectral thm Schauder Basis

$\exists$ complete ONB $u_j \in L^2(\Omega)$ and

$\alpha_j \geq 0 \quad T u_j = \alpha_j u_j$

$\hookrightarrow$ only accumulation pt is 0.

Claim 1 $\quad \alpha_j \neq 0 \;\; \forall j$ we want to take α_j^{-1}

$\alpha_j = 0 \hookrightarrow T u_j = \alpha_j u_j = 0 \to u_j = 0$

because solution operator is injective.

claim 2. $\alpha_j \|u_j\|^2$

$= \alpha_j \langle u_j, u_j \rangle = \langle T u_j, u_j \rangle$

$= \Delta' \langle u_j, u_j \rangle \underset{\text{garding}}{\geq} c_1 \|u_j\|^2_{H_0^m}$

so $\alpha_j > 0$

let $\mu_j = \alpha_j^{-1} - \lambda$

$\lim_{j \to \infty} \mu_j = \infty$ and $\mu_j > -\lambda$

claim $\quad \Delta \langle v, u_j \rangle = \mu_j \langle v, u_j \rangle \;\; \forall v \in H_0^m(\Omega)$

$\Delta' \langle v, u_j \rangle - \lambda \langle v, u_j \rangle$

~~$= \langle v, T u_j \rangle - \lambda \langle v, u_j \rangle$~~

$= \alpha_j^{-1} \Delta' \langle v, \alpha_j u_j \rangle - \lambda \langle v, u_j \rangle$

$= \alpha_j^{-1} \Delta' \langle v, T u_j \rangle - \lambda \langle v, u_j \rangle$

$= \alpha_j^{-1} \langle v, u_j \rangle - \lambda \langle v, u_j \rangle$

$= \mu_j \langle v, u_j \rangle$

$u_j \in L^2$ and $T u_j = \alpha u_j$ output L^2, input H^m?

~~Keep~~ repeating Hell

$\Delta(v, u_j) = \mu_j(v, u_j)$

$\parallel$

$(v, Lu_j) = \mu_j(v, u_j)$

$Lu_j = \mu_j u_j$ By Elliptic Regularity, $u_j \in H^s_{loc}$ $\forall s$

$+ u_j \in C^\infty$

Applic^n of Sobolev Embedding

(Neumann) Poincare Inequality

$\Omega \to$ bounded domain C^1 boundary $\partial\Omega$

$\exists$ constant $C(n, \Omega)$ s.t.

$$\|u - \bar{u}\|_{L^2} \le C \|\nabla u\|_{L^2(\Omega)}$$

$$\bar{u} = \frac{1}{|\Omega|}\int_\Omega u$$

Proof: Renormalize: $v = \frac{u - \bar{u}}{\|u - \bar{u}\|_{L^2}}$

If inequality is not true, $\exists$ a sequence v_k s.t.

$\|v_k\|_{L^2} = 1$ and $\|\nabla v_k\|_{L^2} \to 0$

(recast ineq: $\frac{\|\nabla u\|}{\|u - \bar{u}\|_{L^2}} \ge C'$ is bounded below)

$\therefore$ contra: not bounded below.

$\|\nabla v_k\|_{L^2} \to 0$

v_k is H^1 bounded, By Ray ... / K ... or Compact Sobolev Embedding)

$\therefore$ subsequence converges

$\exists v_{k_j} \to v$ in $L^2(\Omega)$

$\hookrightarrow$ any $H^m_{m \ge 1}$

Claim 1: $\bar{v} = 0$

true since $v_{k_j} \to v$ a.e.

$$\int v_k = \int \frac{u - \bar{u}}{\|u - \bar{u}\|_{L^2}} = \int \frac{u}{\|u - \bar{u}\|} - \int \frac{\bar{u}}{\|u - \bar{u}\|_{L^2}} = \int \frac{u}{\|u - \bar{u}\|} - \bar{u}\int \frac{1}{\|u - \bar{u}\|}$$

$$|\Omega| \frac{\bar{u}}{\|u - \bar{u}\|} - \frac{|\Omega| \bar{u}}{\|u - \bar{u}\|} = 0$$

2. $\|v\|_{L^2(\Omega)} = 1$ since $\|v_{k_j}\|_{L^2} = 1$

3. $v \in H^1(\Omega)$, $\nabla v = 0$ a.e.

4. $\forall \varphi \in C_c^\infty(\Omega)$, $\int_\Omega v \, \partial_j \varphi \overset{DCT}{=} \lim_{j\to\infty} \int_\Omega v_{k_j} \partial_j \varphi$

$= -\lim_{j\to\infty} \int \partial_j v_{k_j} \cdot \varphi \le \|\nabla v_{k_j}\|_{L^2} \|\varphi\|_{L^2} \to 0$

So $\nabla v = 0$ and $v \in H^1(\Omega)$.

Last thing: Check: $v =$ constant a.e.

$\to v = 0$

$\because \int v = 0$ $[\bar{v} = 0]$

Approx by smooth funcn in H^1 and ~~then~~ MVT.

~~contradict that~~ contradicte

$\|v\|_{L^2(\Omega)} = 0$

Not^n: points and sets in Euclidean space

$\mathbb{R}$ ~~denoted~~ denotes real. $\mathbb{C}$: complex

$U \subset \mathbb{R}^n$ then $\overline{U}$: closure and ∂U its boundary

Domain = open set $\Omega \subset \mathbb{R}^n$, not necessarily connected, such that $\partial\Omega = \partial(\mathbb{R}^n - \overline{\Omega})$

That is: Ω has no "interior boundary points"

$x \cdot y = \sum_1^n x_j \overline{y_j} = \sum_{j=1}^n x_j y_j$ for $x, y \in \mathbb{R}$

and $|x| = (x \cdot x)^{1/2}$

Following notation: for spheres and open balls:

if $x \in \mathbb{R}^n$ and $r > 0$

$S_r(x) = \{ y \in \mathbb{R}^n : |x - y| = r \}$,

$B_r(x) = \{ y \in \mathbb{R}^n : |x - y| < r \}$

Measures and Integrals

Integral of a func^n f over a subset Ω of $\mathbb{R}^n$ w.r.t. Lebesgue measure will be denoted by

$\int_\Omega f(x)\,dx$ or simply $\int_\Omega f$, if no sign is ~~to~~ extended over $\mathbb{R}^n$

If S is a smooth hypersurface natural Euclidean measure on S will be denoted by $d\sigma$ $\int_S f(x)\,d\sigma(x)$ or $\int_S f\,d\sigma$ or $\int_S f$

If f and g are f^n whose product is integrable on $\mathbb{R}^n$

$\langle f, g \rangle = \int fg$ $\quad$ $(f, g) = \int f\overline{g}$ complex conj

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The hermitian pairing $\langle f, g \rangle$ will be used only when working directly with Hilbert space L^2 whereas bilinear pairing $\langle f, g \rangle$ more generally.

Multi indices and derivatives

$\alpha = (\alpha_1, \dots, \alpha_n)$ be an n-tuple of non-negative integers. We call α a multi-index. We define

$$|\alpha| = \sum_{1}^{n} \alpha_j, \quad \alpha! = \alpha_1! \alpha_2! \dots \alpha_n!$$

If $x \in R^n$ we set $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$

Short hand: $\partial_j = \partial/\partial x_j$ for derivatives on R^n

Higher order derivatives are then conveniently expressed by multi indices:

$$\partial^\alpha = (\partial/\partial x_1)^{\alpha_1} (\partial/\partial x_2)^{\alpha_2} \dots (\partial/\partial x_n)^{\alpha_n}$$

if $\alpha = 0$ ∂^α is Identity operator

$\partial u = (\partial_1 u, \dots, \partial_n u)$ when u is a differentiable func^n however more common

$$\text{grad } u = (\partial_1 u, \dots, \partial_n u)$$

If $\mu = (\mu_1, \dots, \mu_n)$ is an n-tuple of continuous func^ns on a set $V \subset R^n$ and u is differentiable function, define derivative $\partial_\mu u$ on V by

$$\partial_\mu u(x) = \mu(x) \cdot \text{grad } u(x) = \sum_{1}^{n} \mu_j(x) \partial_j u(x)$$

Function Spaces

If Ω is a subset of $\mathbb{R}^n$, $C(\Omega)$ will denote the space of continuous functions on Ω (with ~~space~~ respect to the relative topology on Ω).
If Ω is open and k is a positive integer, $C^k(\Omega)$ will denote the space of continuous functions possessing continuous derivatives up to order k on Ω, and $C^k(\bar\Omega)$ will denote space of all $u \in C^k(\Omega)$ such that u and $\partial^\alpha u$ ($|\alpha| \le k$) extend continuously to the closure of $\bar\Omega$.
Also we set $C^\infty(\Omega) = \bigcap_1^\infty C^k(\Omega)$
and $C^\infty(\bar\Omega) = \bigcap_1^\infty C^k(\bar\Omega)$

Let $\Omega \subset \mathbb{R}^n$ be open and $0<\alpha<1$. Denote by $C^\alpha(\Omega)$ the space of continuous functions on Ω which satisfy a locally uniform Lipschitz (Hölder) condition with exponent α. That is, $u \in C^\alpha(\Omega)$ means ~~that~~ for any compact $V \in \Omega$ there is a constant $C>0$ ~~st~~ $\forall y \in \mathbb{R}^n$ sufficiently close to 0,

$$\sup_{x \in V} |u(x+y) - u(x)| \le C|y|^\alpha$$

(Note that $C^1(\Omega) \subset C^\alpha(\Omega)$ $\forall \alpha<1$, by the mean value theorem.) If k is +ve integer, $C^{k+\alpha}(\Omega)$: set of all $u \in C^k(\Omega)$ s.t.

$$\partial^\beta u \in C^\alpha(\Omega) \quad \text{for } |\beta| \le k$$

or equivalently for $|\beta| = k$)

The *support* of a function u, denoted by $\mathrm{supp}\ u$, is complement of the largest open set on which $u = 0$.

If $\Omega \subset \mathbb{R}^n$ we denote by $C_0^\infty(\Omega)$ the space of all C^∞ func on $\mathbb{R}^n$ whose support is compact and lies in Ω.

In particular, if Ω is open, such func's vanish near $\partial\Omega$.

The space $C^k(\mathbb{R}^n)$ will be denoted by C^k, likewise C^∞, $C^{k+\alpha}$, C_0^∞

A func said to be *analytic* in Ω if it can be expanded in a power series about every point of Ω. That is, u is analytic on Ω if for each $x \in \Omega$ there exists $r > 0$ such ~~that~~ $\forall\ y \in B_r(x)$

Often denoted by $C^\omega(\Omega)$

$$u(y) = \sum_{|\alpha| \geq 0} \frac{(y-x)^\alpha}{\alpha!} \partial^\alpha u(x)$$

the series being absolutely and uniformly convergent ~~of~~ on $B_r(x)$.

When referring to complex analytic func's we shall use: *holomorphic*

The Schwartz class $S = S(\mathbb{R}^n)$ is the space of all C^∞ functions on $\mathbb{R}^n$ which, together with all ~~of~~ their derivatives die out faster than any power of x at ∞. That is, $u \in S$ iff $u \in C^\infty$ and for all multiindices α, β

$$\sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta u(x)| < \infty$$

B Results from Advanced calculus

A subset S of $\mathbb{R}^n$ is called a hypersurface of class C^k $(1 \leq k \leq \infty)$ if for ~~any~~ every $x_0 \in S$ there is an open set $V \subset \mathbb{R}^n$ containing x_0 and a real valued function $\phi \in C^k(V)$ such ~~that~~ $\operatorname{grad}\phi$ is nonvanishing on $S \cap V$ and

$$S \cap V = \{x \in V : \phi(x) = 0\}$$

In this case, by the implicit function ~~theorem~~ we can ~~show that~~ solve ~~the~~ eqn $\phi(x) = 0$ near x_0 for some coordinate x_i, obtaining

$$x_i = \psi(x_1, \ldots x_{i-1}, x_{i+1}, \ldots x_n)$$

for some C^k function ψ. A neighbourhood of x_0 in S can then be mapped to a piece of the hyperplane $x_n = 0$ by the C^k transformation:

$C: x_i = \psi(x')$

$$x \rightarrow (x', x_i - \psi(x')) \quad (x' = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n))$$

The same neighbourhood can also be represented in parametric form as ~~the~~ image of an open set in $\mathbb{R}^{n-1}$ (with coordinate x') under the map

$$x' \rightarrow (x_1, \ldots x_{i-1}, \psi(x'), x_{i+1}, \ldots x_n).$$

x' may be ~~thought~~ of as giving local coordinates near x_0. Similar consider^ns apply when "C^k" is replaced by analytic. With S, V, ϕ as above, vector $\operatorname{grad}\phi(x)$ is $\perp$ to S at x $\forall x \in S \cap V$. We shall always suppose ~~that~~ S is oriented. That is, that we have made a choice of ~~a~~ unit vector $\nu(x)$ for each $x \in S$, varying continuously with x, which is perpendicular to S at x. $\nu(x)$ will be called normal to S at x. clearly on $S \cap V$, we have

$$\nu(x) = \pm \frac{\operatorname{grad}\phi(x)}{|\operatorname{grad}\phi(x)|}$$

$\therefore \nu(x)$ is a C^{k-1} function on S.

If S is ~~the~~ boundary of a domain Ω, always choose orient^n so ~~that~~ ν points out of Ω!

If u is a differentiable function defined near S, we can define the normal derivative of u on S by

$$\partial_\nu u = \nu \cdot \operatorname{grad} u$$

Compute Normal derivative on the sphere $S_r(y)$

Since lines through the centre of sphere are $\perp$ to sphere, we have

$$\nu(x) = \frac{x-y}{r}, \quad \partial_\nu = \frac{1}{r} \sum_1^n (x_j - y_j)\partial_j \text{ on } S_r(y)$$

We will use the following propos^n several time

(0.2) Proposition: Let S be a compact oriented hypersurface of class C^k, $k \geq 2$. There is a neighbourhood V of S in $\mathbb{R}^n$ and a number $\varepsilon > 0$ such that the map

$$F : (x, t) \to x + t\nu(x)$$

is a C^{k-1} homeomorphism of $S \times (-\varepsilon, \varepsilon)$ onto V

Proof: F is clearly C^{k-1}. Moreover for each $x \in S$ ~~its~~ Jacobian matrix (with respect to local coordinates on $S \times \mathbb{R}$) at $(x, 0)$ is non-singular since ν is normal to S. Hence by inverse mapping thm, F can be inverted on a neighbourhood W_x of each $(x, 0)$ to yield a C^{k-1} map

$$F_x^{-1} : W_x \to (S \cap W_x) \times (-\varepsilon(x), \varepsilon(x))$$

for some $\varepsilon(x) > 0$. Since S is compact, we can choose $x_1, x_2, \ldots, x_N \in S$ such that the W_{x_j} cover S and the maps $F_{x_j}^{-1}$ patch together to yield a C^{k-1} inverse of F from a neighbourhood V ~~of~~ of S to $S \times (-\varepsilon, \varepsilon)$ where $\varepsilon = \min \varepsilon(x_j)$

The neighbourhood V in propos^n is called tubular neighborhood of S. It will be convenient to extend the definition of normal derivative to the whole tubular ~~ne~~ neighborhood of S. Namely, if u is a diff func^n on V, for $x \in S$ and $-\varepsilon < t < \varepsilon$ we set

$$\partial_\nu u(x + t\nu(x)) = \nu(x) \cdot \operatorname{grad} u(x + t\nu(x))$$

For our purposes a vector field will be simply an $\mathbb{R}^n$ valued function on a subset of $\mathbb{R}^n$. If $\mu = (\mu_1, \ldots, \mu_n)$ is a differentiable vector field on an open set $\Omega \subset \mathbb{R}^n$, we defined <u>divergence</u> $\operatorname{div} \mu$ on Ω by

$$\operatorname{div} \mu = \sum_1^n \partial_j \mu_j$$

In this notation form of general Stokes formula:

<u>(0.4) The Divergence Theorem</u> Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^1 boundary $S = \partial \Omega$ and let μ be a vector field of class C^1 on $\bar{\Omega}$. Then

$$\int_S \mu(y) \cdot \nu(y) \, d\sigma(y) = \int_\Omega \operatorname{div} \mu(x) \, dx$$

Every $x \in \mathbb{R}^n - \{0\}$ can be written uniquely as $x = ry$ with $r > 0$ and $y \in S_1(0)$ - namely, $r = |x|$ and $y = x/|x|$. The formula $x = ry$ is called polar coordinate rep. of x. Lebesgue measure in polar coord is given by

$$dx = r^{n-1} \, dr \, d\sigma(y)$$

where $d\sigma$ is surface measure on $S_1(0)$

For ex: if $0 < a < b < \infty$ and $\lambda \in \mathbb{R}$, we have

$$\int_{a<|x|<b} |x|^\lambda \, dx = \int_{S_1(0)} \int_a^b r^{n-1+\lambda} \, dr$$

$$= \begin{cases} \omega_n (n+\lambda)^{-1} (b^{n+\lambda} - a^{n+\lambda}) & \text{if } \lambda \neq -n \\ \omega_n \log b/a & \text{if } \lambda = -n \end{cases}$$

integrand reduces to r^{-1} ...

where ω_n is area of $S_1(0)$.

Immediate consequence:

<u>(0.5) Proposition</u>: The func^n $x \to |x|^\lambda$ is integrable on a neighbourhood of 0 iff $\lambda > -n$ and integrable outside a neighborhood of 0 iff $\lambda < -n$

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* Most imp definite integral in Math.

(0.6) Prop^n $\int_{\mathbb{R}^n} e^{-\pi|x|^2}\,dx = 1$

Proof let $I_n = \int_{\mathbb{R}^n} e^{-\pi|x|^2}\,dx$

$$e^{-\pi|x|^2} = e^{-\pi\sum_1^n x_j^2} = \prod_1 e^{-\pi x_j^2}$$

Fubini's thm: $I_n = (I_1)^n$ or equivalently

$I_n = (I_2)^{n/2}$. But in Polar coord

$$I_2 = \int_0^{2\pi}\int_0^\infty e^{-\pi r^2}\, r\,dr\,d\theta$$

$$= 2\pi \int_0^\infty r e^{-\pi r^2}\,dr$$

$$= \pi\int_0^\infty e^{-\pi s}\,ds = \pi/\pi = 1$$

This trick works because we know area of $S_1(0)$ in $\mathbb{R}^2$ is 2π. But now we can turn it around to compute ω_n of $S_1(0)$ in $\mathbb{R}^n$ for any n.

Recall ~~that~~ gamma func^n $\Gamma(s)$ is defined for $\operatorname{Re} s > 0$ by

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\,dt$$

$\Gamma(1+s) = s\Gamma(s)$
$\Gamma(1) = 1$
$\Gamma(2+1) = 2\Gamma(2)$
$\Gamma(3) = 2\times 1$
$\Gamma(4) = 3\times 2\times 1$
$\Gamma(k) = (k-1)!$

verify $\Gamma(s+1) = s\Gamma(s)$ $\quad \Gamma(1) = 1$

$\Gamma(1/2) = \sqrt{\pi}$

here if k is +ve integer

$$\Gamma(k) = (k-1)! \qquad \Gamma(k/2) = \left(\frac{k-1}{2}\right)\left(\frac{k-3}{2}\right)\cdots\left(\frac{1}{2}\right)\sqrt{\pi}$$

(k odd)

(0.7) Propos^n Area of $S_1(0)$ in $\mathbb{R}^n$ is

$$\omega_n = 2\pi^{n/2}/\Gamma(n/2)$$

Proof we integrate $e^{-\pi|x|^2}$ in polar coord

set $s = \pi r^2$

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$$1 = \int e^{-\pi|x|^2}dx = \int_{S_1(0)}\int_0^\infty e^{-\pi r^2} r^{n-1}\,dr\,d\sigma$$

$$= \omega_n \int_0^\infty e^{-\pi r^2} r^{n-1}\,dr = \left(\frac{\omega_n}{2\pi^{n/2}}\right)\int_0^\infty e^{-s} s^{n/2-1}\,ds$$

$$= \omega_n \frac{\Gamma(n/2)}{2\pi^{n/2}}$$

ω_n is always rational multiple of an integer power of π

(0.8) Corollary: The volume of $B_1(0)$ in $\mathbb{R}^n$ is $\frac{2\pi^{n/2}}{n\Gamma(n/2)}$

Proof $\int_{B_1(0)} dx = \omega_n \int_0^1 r^{n-1}\,dr = \omega_n/n$

(0.9) Corollary for any $x \in \mathbb{R}^n$ and $r>0$, the area of $S_r(x)$ is $r^{n-1}\omega_n$, and volume of $B_r(x)$ is $r^n\omega_n/n$.

C. Convolutions

General Thm about Integral operators on a measure space (S, μ) which deserves to be more widely known. In our applications, S will be either $\mathbb{R}^n$ or a smooth hypersurface in $\mathbb{R}^n$.

(0.10) Generalized Young's Inequality: Let (S, μ) be a measure space and let $1 \le p \le \infty$ and $C > 0$. Suppose K is a measurable func on $S \times S$ such that

$$\int_S |K(x,y)|\,d\mu(y) \le C \quad \text{for all } x \in S \text{ and}$$

$$\int_S |K(x,y)|\,d\mu(x) \le C \quad \forall\, y \in S, \text{ and suppose}$$

$f \in L^p(S)$. Then the func Tf defined by

$$Tf(x) = \int K(x,y) f(y)\,d\mu(y)$$

is well defined almost everywhere and is in $L^p(S)$, and

$$\|Tf\|_p \le C\|f\|_p$$

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Proof If $p=\infty$, the hypothesis $\int_S |K(x,y)|\,d\mu(x) \le C$ is superfluous and conclusion is obvious.

If $p<\infty$, let q be the conjugate exponent, i.e. $\frac{1}{p}+\frac{1}{q}=1$

then by Holder's Inequality

$$|Tf(x)| \le \left(\int_S |K(x,y)|\,d\mu(y)\right)^{1/q}\left(\int_S |K(x,y)|\,|f(y)|^p\,d\mu(y)\right)^{1/p}$$

$$\le C^{1/q}\left(\int_S |K(x,y)|\,|f(y)|^p\,d\mu(y)\right)^{1/p}.$$

Raise both ~~sides~~ to pth power and integrate, we see by Fubini's then ~~that~~:

$$\int |Tf(x)|^p\,d\mu(x) \le C^{p/q}\int_S\int_S |K(x,y)|\,|f(y)|^p\,d\mu(y)\,d\mu(x)$$

Fubini

$$\le C^{p/q+1}\int_S |f(y)|^p\,d\mu(y)$$

taking pth roots

$$\|Tf\|_p \le C^{1/p+1/q}\|f\|_p = C\|f\|_p$$

Let f and g be locally integrable func^ns on $\mathbb{R}^n$.

The convolution $f*g$ of f and g is defined by

$$f*g(x) = \int f(x-y)g(y)\,dy$$

$\int \ldots dy$ x is const $x-y=t$ $-dy=dt$ $\int f(t)g(x-t)\,dt$ used

$$= \int f(y)g(x-y)\,dy = g*f(x)$$

change of var $y \to x-y$

Basic Thms on existence of convolutions is the following:

(0.11) Young's Inequality. If $f\in L^1$ and $g\in L^p$ $(1\le p\le\infty)$ ~~then~~ $f*g\in L^p$ and $\|f*g\|_p \le \|f\|_1\|g\|_p$

Proof Apply (0.10) with $S=\mathbb{R}^n$ and $K(x,y)=f(x-y)$

i.e. $f(x-y)$ is a measurable func^n on $\mathbb{R}^n\times\mathbb{R}^n$ ($f\in L^1$) such that $\int_{\mathbb{R}^n} |f(x-y)|\,d\mu y = \|f\|_1 \le C$ $\forall x\in\mathbb{R}^n$ assuming $y\in\mathbb{R}^n$

and $g\in L^p$ then $Tg(x) = \int_{\mathbb{R}^n} f(x-y)\,g(y)\,d\mu(y)$ is in $L^p(\mathbb{R}^n)$ and $\|Tg\|_p \le \|f\|_1\,\|g\|_p$

Remark Obvious from Holder's Inequality that if $f \in L^p$ and $g \in L^q$ where $\frac{1}{p} + \frac{1}{q} = 1$ then $f * g \in L^\infty$

and $\|f * g\|_\infty \leq \|f\|_q \|g\|_p$

From Riesz convexity theorem (cf. Stein-Weiss [ ]) one can deduce following generalization of Young's inequality

suppose $1 \leq p, q, r \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$

If $f \in L^q$ and $g \in L^p$ then $f * g \in L^r$ and $\|f * g\|_r \leq \|f\|_q \|g\|_p$

Next thm underlies one of the most important uses of convolutions. We need a technical lemma before

If f is a function on $\mathbb{R}^n$ and $x \in \mathbb{R}^n$, we define the function f_x by $f_x(y) = f(x+y)$

(0.12) Lemma: Suppose: $1 \leq p < \infty$ and $f \in L^p$. Then

$\lim_{x \to 0} \|f_x - f\|_p = 0$

Proof If g is continuous with compact support, then g is uniformly continuous, so $g_x \to g$ uniformly as $x \to 0$ since g_x and g are supported in a common compact set for $|x| \leq 1$, it follows that $\|g_x - g\|_p \to 0$. Now given $f \in L^p$ and $\varepsilon > 0$, choose a continuous g with compact support such that $\|f - g\| < \frac{\varepsilon}{3}$ Then also $\|f_x - g_x\|_p < \varepsilon/3$ so

$$\|f_x - f\|_p \leq \|f_x - g_x\|_p + \|g_x - g\|_p + \|g - f\|_p < \|g_x - g\|_p + \frac{2\varepsilon}{3}$$

But for x sufficiently small, $\|g_x - g\|_p < \varepsilon/3$ so $\|f_x - f\|_p < \varepsilon$

Remark: This result is false for $p = \infty$. Indeed, the condition that $f_x \to f$ uniformly is just the condition that f be uniformly continuous

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(0.13) Theorem Suppose $\phi \in L^1$ and $\int \phi(x)\,dx = a$. For each $\varepsilon > 0$, define the function ϕ_ε by

$$\phi_\varepsilon(x) = \varepsilon^{-n}\phi\left(\frac{x}{\varepsilon}\right)$$

Suppose $f \in L^p$, $1 \le p \le \infty$. Then if $p < \infty$,

$$f * \phi_\varepsilon \to af \text{ in the } L^p \text{ norm as } \varepsilon \to 0$$

If $f \in L^\infty$ and f is uniformly continuous on a set V, then $f * \phi_\varepsilon \to af$ uniformly on V as $\varepsilon \to 0$.

Proof: By the change of variables $x \to \varepsilon x$ we see that $\int \phi_\varepsilon(x)\,dx = a$ for all $\varepsilon > 0$. Hence

$$f * \phi_\varepsilon(x) - af(x) = \int [f(x-y) - f(x)]\,\phi_\varepsilon(y)\,dy$$

$$= \int [f(x-\varepsilon y) - f(x)]\,\phi(y)\,dy$$

If $f \in L^p$ and $p < \infty$, we apply Δ ineq for integrals & integral is limit of sums to obtain:

$$\|f * \phi_\varepsilon - af\|_p \le \int \|f_{-\varepsilon y} - f\|_p\, |\phi y|\,dy$$

But $\|f_{-\varepsilon y} - f\|$ is bounded by $2\|f\|_p$ and tends to 0 as $\varepsilon \to 0$ by Lemma (0.12). Result thus follows from Lebesgue dominated convergence ~~theorem~~ theorem.

On the other hand, suppose $f \in L^\infty$ and f is uniformly cont. on V. Given $\delta > 0$, choose a compact set W so that

$$\int_{\mathbb{R}^n - W} |\phi| < \delta \quad \text{then} \quad \sup_{x\in V} |f * \phi_\varepsilon(x) - af(x)| \le$$

$$\sup_{x\in V,\, y\in W} |f(x-\varepsilon y) - f(x)| \int_W |\phi| + 2\|f\|_\infty \delta$$

1st term on RHS $\to 0$ as $\varepsilon \to 0$ and δ is arbit. so $f * \phi_\varepsilon$ tends uniformly to f on V.

If $\phi \in L^1$ and $\int \phi(x)\,dx = 1$, the family of functions $\{\phi_\varepsilon\}$ defined ~~by~~ above is called an approximation to the identity.

$$f * \phi_\varepsilon(x) - af(x)$$

$$\int \phi(y)\,dy = a$$

$$\int \underset{\substack{\uparrow \\ \text{const}}}{f(x)}\, \phi(y)\,dy - af(x)$$

$$= \int f(x-y)\,\phi_\varepsilon(y)\,dy - \int f(x)\,\phi_\varepsilon(y)\,dy$$

$$= \int (f(x-y) - f(x))\,\phi_\varepsilon(y)\,dy$$

$$\frac{1}{\varepsilon^n}\,\phi(y/\varepsilon)\,dy$$

$$= \int (f(x-\varepsilon t) - f(x))\,\phi(\cancel{t})\,dt \qquad y/\varepsilon \to t$$

$$dy \to \varepsilon^n dt$$

$$= \int (f(x-\varepsilon y) - f(x))\,\phi(y)\,dy$$

$$\partial^\alpha (f * \phi) \underset{\text{wrt } x^i}{=} \partial^\alpha \int f(y)\,\phi(x-y)\,dy$$

$$\underset{\text{conv.}}{\overset{\text{dominant}}{=}} \int \partial^\alpha f(x)\,\phi(x-y)\,dy$$

$$= \int f(y)\,\partial^\alpha \phi(x-y)\,dy$$

$$\boxed{\partial^\alpha (f * \phi) = f * \partial^\alpha \phi}$$

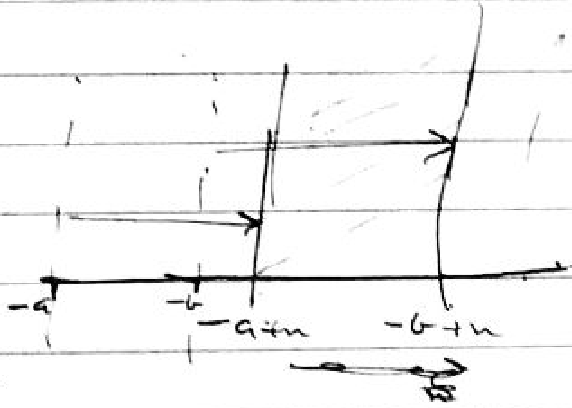
$$\int_{y=a}^{y=b} f(x-y)\, g(y)\, dy$$

$$x - y = t$$

$$dy = -dt$$

$y=b \to t = x-b$
$y=a \to t = x-a$

$$-\int_{x-a}^{x-b} f(t)\, g(x-t)\, dt$$



$$t = y$$

$$-\int_{-a+x}^{-b+x} f(y)\, \underbrace{g(x-y)}\, dy \longrightarrow \int F(y, x-y)\, dy$$

$$-(-1)\int_{+a}^{+b} f(y)\, g(x-y)\, dy$$

$\therefore$ $\cancel{f * g = \int f(y)\, g(x-y)}$

$$f * g(x) = \int f(y)\, g(x-y)\, dy = \int f(x-y)\, g(y)\, dy$$

What makes these useful is that by choosing ϕ appropriately we can get the functions $f * \phi_\epsilon$ ~~to~~ have nice properties. In particular:

(0.14) **Theorem** If $f \in L^p$ $(1 \le p \le \infty)$ and ϕ is in the Schwartz class $\mathcal{S}$, then $f*\phi \in C^\infty$ and $\partial^\alpha(f*\phi) = f*\partial^\alpha\phi$ for all α

Proof: If $\phi \in \mathcal{S}$, then every derivative $\partial^\alpha\phi$ is in L^q for every q, in particular when $\frac{1}{p}+\frac{1}{q} = 1$

Thus by Hölder's Inequality the integral

$$f * \partial^\alpha \phi(x) = \int f(y)\partial^\alpha\phi(x-y)\,dy$$

converges absolutely and uniformly on $\mathbb{R}^n$.

Differentiation can thus be interchanged with integration and we can conclude $\partial^\alpha(f*\phi) = f*\partial^\alpha\phi$

We can get more precise results by taking $\phi \in C_0^\infty$. In ~~that~~ case we need only assume ~~that~~ f is locally integrable for $f*\phi$ to be well defined, and the same argument shows ~~that~~ $f*\phi \in C^\infty$

Existence of functions in C_0^∞ is not trivial. We pause for constructing these.

First define the function f on $\mathbb{R}$ by

$$f(t) = \begin{cases} e^{1/1-t^2} & (|t|<1) \\ 0 & (|t| \ge 1) \end{cases}$$

then $f \in C_0^\infty(\mathbb{R})$ so ~~that~~ $\psi(x) = f(|x|^2)$ is a non negative C^∞ function on $\mathbb{R}^n$ supported in $\overline{B_1(0)}$. In particular

$\int\psi > 0$, so $\phi = \psi / \int\psi$ is a func^n in C_0^∞ with $\int\phi = 1$.

Now there can be lots of functions in C_0^∞

(0.15) **Lemma** If f is supported in $V \subset \mathbb{R}^n$ and g is supported in $W \subset \mathbb{R}^n$, then $f*g$ is supported in $\{x+y : x \in V, y \in W\}$.

Proof: Left as exercise.

(0.16) Theorem C_0^∞ is dense in L^p for $1 \le p < \infty$

Proof Choose $\phi \in C_0^\infty$ with $\int \phi = 1$, and set $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(x/\varepsilon)$. If $f \in L^p$ has compact support, it follows from (0.14) and (0.15) ~~that~~ $f * \phi_\varepsilon \in C_0^\infty$ and from (0.13) ~~that~~ $f * \phi_\varepsilon \to f$ in L^p. But L^p functions with compact support are dense in L^p, so we are done.

Another useful construction.

(0.17) Theorem Let $V \subset \mathbb{R}^n$ be compact and $\Omega \subset \mathbb{R}^n$ be open, and assume $\bar{V} = \Omega$. Then there exists $f \in C_0^\infty(\Omega)$ such ~~that~~ $f = 1$ on V and $0 \le f \le 1$ everywhere.

Proof Let $\delta = \inf\{|x-y| : x \in V, y \notin \Omega\}$. By our assumptions on V and Ω, $\delta > 0$. Let
$U = \{x : |x-y| < \delta/2 \text{ for some } y \in V\}$. Then $V \subset U$ and $\bar{U} \subset \Omega$. Let χ be characteristic function of U, and choose a non-negative $\phi \in C_0^\infty(B_{\delta/2}(0))$ such ~~that~~ $\int\phi = 1$. Then we can take $f = \chi * \phi$; ~~the~~ simple verification is left to ~~the~~ reader.

We can now prove existence of "partitions of unity". We state the following results only for compact sets, which is all we need, but they are true more generally.

(0.18) Lemma Let $\Omega \subset \mathbb{R}^n$ be compact and let $V_1 \ldots V_N$ be open sets with $\Omega \subset \bigcup_1^N V_j$. Then there exist open sets $W_1, W_2, \ldots, W_N$ with $\overline{W_j} \subset V_j$ and $\Omega \subset \bigcup_1^N W_j$.

Proof For each $\varepsilon > 0$ let V_j^ε be the set of points in V_j whose distance from $\mathbb{R}^n \setminus V_j$ is greater than ε. Clearly V_j^ε is open and $\overline{V_j^\varepsilon} \subset V_j$. We claim ~~that~~ $\Omega \subset \bigcup_1^N V_j^\varepsilon$ if ε is sufficiently small. Otherwise

for each $\varepsilon > 0$ there exists $x_\varepsilon \in \overline{\Omega} - \cup_1^N V_j^\varepsilon$
Since $\overline{\Omega}$ is compact, the x_ε have an accumulation point $x \in \overline{\Omega}$ as $\varepsilon \to 0$. But then $x \in \overline{\Omega} - \cup_1^N V_j$, which is absurd.

(Thm) Let $\overline{\Omega} \subset \mathbb{R}^n$ be compact and let $V_1, \ldots V_N$ be bounded open sets such that $\overline{\Omega} \subset \cup_1^N V_j$.
Then there exists functions $\zeta_1, \ldots \zeta_N$ with $\zeta_j \in C_0^\infty(V_j)$ such that $\sum_1^N \zeta_j = 1$ on $\overline{\Omega}$.

Proof: Choose open sets $W_1, \ldots, W_N$ as in Lemma (0.18), and choose $\phi_j \in C_0^\infty(V_j)$ with $0 \le \phi_j \le 1$ and $\phi_j = 1$ on W_j. (This is possible by Thm (0.17) since $\overline{W_j}$ is compact.) Then $\sum_1^N \phi_j \ge 1$ on $\overline{\Omega}$ so we can take

$$\zeta_j = \phi_j / \sum_1^N \phi_i$$

The collection of functions $\{\zeta_j\}$ is called partition of unity on $\overline{\Omega}$ subordinate to the covering $\{V_j\}$

D) Fourier Transform

Rapid intro. For more extensive: Stein-Weiss
If $f \in L^1(\mathbb{R}^n)$, its Fourier transform $\hat{f}$ is bounded function on $\mathbb{R}^n$ defined by:

$$\hat{f}(\xi) = \int e^{-2\pi i x \cdot \xi} f(x)\, dx$$

(makes FT both an isometry on L^2 and an algebra homomorphism from L^1 (with convol^n) to L^∞ (with pt. wise multiplic^n)
for all ξ, $\|\hat{f}\|_\infty \le \|f\|_1$

Moreover: Thm If $f, g \in L^1$ ~~then~~ $(f * g)^\wedge = \hat{f}\hat{g}$
Proof: Simple app of Fubini's thm

$$(f*g)^\wedge(\xi) = \iint e^{-2\pi i x \cdot \xi} f(x-y)\, g(y)\, dy\, dx$$
$$= \int e^{-2\pi i (x-y)\cdot\xi} f(x-y) (e^{-2\pi i y \cdot \xi} g(y)\, dy)\, dx$$
$$= \hat{f}(\xi) \int e^{-2\pi i y \cdot \xi} g(y)\, dy = \hat{f}(\xi)\, \hat{g}(\xi)$$

Consider its restriction to the Schwartz Class $\mathcal{S}$

(0.21) Prop^n If $f \in \mathcal{S}$, then $\hat{f} \in C^\infty$ and $\partial^\beta \hat{f} = \hat{g}$ where $g(x) = (2\pi i x)^\beta f(x)$

Proof Differentiate under the integral sign

(0.22) Prop^n If $f \in \mathcal{S}$, ~~then $\hat{f} \in C^\infty$~~ $(\partial^\beta f)^\wedge(\xi) = (2\pi i \xi)^\beta \hat{f}(\xi)$

Proof $(\partial^\beta \hat{f})(\xi) = \partial^\beta \left(\int e^{-2\pi i x \cdot \xi} f(x)\, dx \right)$

$$= \int_{-\infty}^{\infty} (-2\pi i x)^\beta e^{-2\pi i x \cdot \xi} f(x)\, dx$$

$$\overset{\text{By parts}}{=} \left(\int f(x)(-2\pi i x)^\beta \int e^{-2\pi i x \xi} dx \right)$$

$$- \left(\int \frac{\partial f(x)(-2\pi i x)^\beta}{\partial x} \times \int e^{-2\pi i x \xi} dx \right)$$

$$\overset{f \to 0 \text{ as } x \to \infty}{=} (0) - \int \frac{\partial f(x)(-2\pi i x)^\beta}{\partial x} \frac{e^{-2\pi i x \xi}}{(-2\pi i)\xi} dx$$

$$= -\int \frac{\partial f}{\partial x}(-2\pi i x)^\beta + \beta \frac{(-2\pi i x)^{\beta-1} (-2\pi i) f(x)}{(-2\pi i)\xi} \int e^{-2\pi i x \xi} dx$$

$$= \left(e^{-2\pi i x \cdot \xi} \right) \int f(x)(-2\pi i x)^\beta dx \Big|_{-\infty}^{\infty}$$

$$- \int (-2\pi i \xi) e^{-2\pi i x \xi} \left(\int f(x)(-2\pi i x)^\beta dx \right) dx$$

$$(\partial^\beta f)^\wedge(\xi) = \int e^{-2\pi i x \xi} \partial^\beta f(x)\, dx$$

$$= \left(e^{-2\pi i x \xi} \int \partial^\beta f(x) \right)_{-\infty}^{\infty} - \int (2\pi i \xi) e^{-2\pi i x(\xi)} \int \partial^\beta f(x) dx$$

$$(\partial^\beta f)^\wedge(\xi) \overset{\text{all } \partial^\beta \text{ vanishes}}{=} 0 - (2\pi i \xi)(\partial^{\beta-1} f)^\wedge(\xi)$$

$\xrightarrow{\text{recursion}}$

$$(\partial^\beta f)^\wedge(\xi) = (-1)^\beta (-2\pi i \xi)^\beta \partial^0 \hat{f}(\xi)$$

$$(\partial^\beta f)^\wedge(\xi) = (2\pi i \xi)^\beta \hat{f}(\xi)$$

Note: $(\partial^\beta f)\hat{}\,\xi = \int \left(\frac{\partial}{\partial x}\right)^\beta f(x)\, e^{-2\pi i \xi x}\,dx = (2\pi i \xi)^\beta \hat{f}$

whereas $\partial^\beta \hat{f} = \left(\frac{\partial}{\partial k}\right)^\beta (\hat{f}) = \left(\frac{\partial}{\partial k}\right)^\beta \int e^{-i2\pi k x} f(x)\,dx$

$= ((-2\pi i x)^\beta f)\hat{}$

0.23 Prop^n: If $f \in \mathcal{S}$ then $\hat{f} \in \mathcal{S}$

By 0.21, 0.22 $\rightarrow$ $\partial^\beta \xi^\alpha \hat{f} = \hat{g}$ where

$\cancel{g(x) = \partial^{\beta-1}\left(\left(\frac{\partial}{\partial \xi}\xi^\alpha\right)\hat{f} + \partial \hat{f}\,\xi^\alpha\right)}$

$\cancel{= \partial^{\beta-1}\left(\alpha \xi^{\alpha-1}\hat{f} + ((-2\pi i x)^\beta f)\hat{}\,\alpha\xi^\alpha\right)}$

$\partial^\beta(\xi^\alpha \hat{f}) = \cancel{\partial^\beta \xi^\alpha \int f(x) e^{i2\pi\xi x}\,dx}$

$\cancel{= \partial^\beta \int (f(x)\xi^\alpha) e^{i2\pi\xi x}\,dx}$

$\cancel{\xi^\alpha = e^{\ln \xi^\alpha}} = e^{\alpha \ln \xi} \rightarrow \cancel{\partial^\beta e^{\alpha \ln \xi} = (\alpha) e^{\alpha \ln \xi}\frac{1}{\xi}}$

$\cancel{= \partial^\beta \int f(x) e^{-i2\pi\xi x + \alpha \ln \xi}\,dx}$

$\rightarrow \partial^\beta \cancel{(\hat{F}) = \partial(-2\pi i x)^\beta} \hat{F}$

~~late~~ 0.22: $(\partial^\beta f)\hat{}\,(\xi) = (2\pi i \xi)^\beta \hat{f}(\xi)$

$\therefore (2\pi i)^\beta \xi^\beta = \dfrac{(\partial^\beta f)\hat{}\,(\xi)}{\hat{f}(\xi)}$

$\partial^\beta\left(\dfrac{(\partial^\alpha f)\hat{}\,(\xi)}{(2\pi i)^\alpha \hat{f}(\xi)} \times \hat{f}(\xi)\right) = \partial^\beta\left((\partial^\alpha f)\hat{}\,(\xi)\right)$

$= \left(\dfrac{(-2\pi i x)^\beta}{(2\pi i)^\alpha}(\partial^\alpha f)\right)\hat{}\,\xi$

$g(x) = (-1)^{|\beta|}(2\pi i)^{|\beta|-|\alpha|} \times x^\beta \partial^\alpha f(x)$

then $\hat{g}$ gives ~~the~~ required result

(0.24) Riemann–Lebesgue Lemma: $f \in L^1$ then $\hat{f}$ is cont. and tends to 0 at ∞.

Proof

(0.25) Thm: $f(x) = e^{-\pi a|x|^2}$ where $a > 0$
then $\hat{f}(\xi) = a^{-n/2}\, e^{-\pi|\xi|^2/a}$

change of var: $x \to \frac{x}{\sqrt{a}}$

Fubini's Thm: exp → sum to products

$$\hat{f}(\xi) = \int e^{-2\pi i x\cdot\xi - \pi|x|^2}\,dx$$

$$= \prod_1^n \int \exp(-2\pi x_j\xi_j - \pi x_j^2)\,dx$$

for $n=1$ $\int e^{-2\pi i x\xi - \pi x^2}\,dx = \int e^{-\pi(x^2 + 2ix\xi - \xi^2)}e^{-\pi\xi^2}\,dx$

$$= e^{-\pi\xi^2}\int e^{-(\pi(x+i\xi)^2)}\,dx$$

$e^{-\pi z^2}$ entire Holomorph.

→ Cauchy: shift integr contour from $\operatorname{Im}(z)=0$ to $\operatorname{Im} z = -\xi$

Together with 0.6 → $\int_{\mathbb{R}^n} e^{-\pi|x|^2}\,dx = 1$

$$= e^{-\pi\xi^2}\left(\int e^{-\pi(x+i\xi)^2}\,dx\right) = e^{-\pi\xi^2}\int e^{-\pi x^2}\,dx$$

$$= e^{-\pi\xi^2}$$

2.2. $\hat{f}(\xi)$ is radial when $f \in \mathcal{S}(\mathbb{R})$ and f is radial.

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-i\langle \xi, x\rangle}\, dx$$

$$\hat{f}(T\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-i\langle \xi, T^* x\rangle}\, dx$$

$$\cancel{= \int_{\mathbb{R}^n} f}$$

$z = T^* x$) for unitary T.
$dz = dx$

$$\hat{f}(T\xi) = \int_{\mathbb{R}^n} f(Tz)\, e^{-i\langle \xi, z\rangle}\, dz$$

radial func^n

$$= \int_{\mathbb{R}^n} f(z)\, e^{-i\langle \xi, z\rangle}\, dz$$

$$= \int_{\mathbb{R}^n} f(z)\, e^{-i\langle \xi, z\rangle}\, dz$$

$\hat{f}(T\xi) = \hat{f}(\xi) \longrightarrow \hat{f}(\xi)$ is radial.

2.3) $f = \chi_{[x^2+y^2 \le 1]}$

$$\hat{f}(\xi) = \int_{B_1(0)} e^{-i\langle \xi, x\rangle}\, dx$$

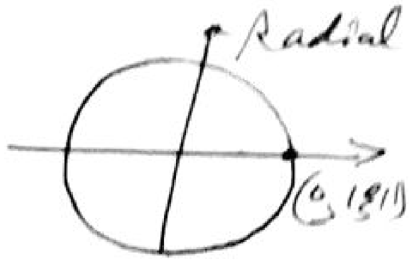
$$\hat{f}(|\xi|) = \int_{B_1(0)} e^{-i x_2 \xi_2}\, dx$$

$$= \int_{0}^{1} \int_{0}^{\sqrt{1-x_1^2}} e^{-i x_2 \xi_2}\, dx_2\, dx_1$$

$$= \int_0^1 \left[\frac{e^{-ix_2\xi_2}}{i\xi_2} \right]_0^{\sqrt{1-x_1^2}}$$

$=$

$$\int_{B_1(0)} e^{-i(\xi_1 x_1 + \xi_2 x_2)}\, dx$$



$$\int_0^1 \int_0^{2\pi} e^{-i|\xi_2| r \cancel{\cos}\sin\theta}\, r\, dr\, d\theta$$

$$e^{-i|\xi_2| r \sin\theta}\, d\theta$$

$$\int_0^1 r\cos(\theta|\xi_2| r \sin\theta)\, dr - \underbrace{i r \sin(\theta|\xi_2| r\sin\theta)\, d\theta}_{0}$$

J

1.2. Heat eqn $\partial_t \int u^2$

$$= 2\int u u_t$$

$$= \int 2u\Delta u = -2\int|\nabla u|^2 \le 0$$

$$\int |\nabla u|^2 = -\int \Delta u\, u$$

$$\partial_t \int |\nabla u|^2 = -\int \Delta u_t u - \int \Delta u\, u_t$$

$$= \int -u_t \Delta u + \int -\Delta u\, u_t$$

$$= -2\int |\Delta u|^2 \le 0$$

$\psi(t) = |t|^{p-2} t$

$\psi'(t) =$

Folland

E. Distributions

convergence in C_0^∞

$\{\phi_j\}$ sequence in C_0^∞, ϕ_j converges to $\phi \in C_0^\infty$

$\phi_j \longrightarrow \phi$ in C_0^∞ if the ϕ_js all have common compact support and $\|\partial^\alpha(\phi_j - \phi)\|_\infty \to 0$ $\forall \alpha$.

Modern: for each bounded set Ω, the space $C_0^\infty(\Omega)$ is a Frechet space under the family of norms:

$\|u\|_{(\alpha)} = \|\partial^\alpha u\|_\infty$, and we put strict inductive limit topology on $C_0^\infty = C_0^\infty(\mathbb{R}^n)$

A distribution is a linear functional on C_0^∞ which is continuous wrt the above notion of convergence

We denote the number obtained by applying a distrib^n u to $\phi \in C_0^\infty$ by $\langle u, \phi \rangle$

Put a weak topology on the space of distributions:

$u_j \to u$ as distributions if $\langle u_j, \phi \rangle \to \langle u, \phi \rangle$ $\forall \phi \in C_0^\infty$

Every locally integrable funct^n u on $\mathbb{R}^n$ can be regarded as a distribution ~~$\langle u_j, \phi \rangle \to u, \phi$~~

$$\langle u, \phi \rangle = \int u \phi$$

Continuity follows from Lebesgue dominated convergence theorem

This correspondence is one to one when we regard two functions as the same when they are equal almost everywhere. Thus distributions can be regarded as "generalized functions".

more generally any locally finite measure μ defines a distribution by the formula $\langle \mu, \phi \rangle = \int \phi \, d\mu$

In particular, if we take μ to be pt. mass at 0 we obtain granddaddy of all distributions, the Dirac ~~delta~~ "δ" function:

$$\langle \delta, \phi \rangle = \phi(0)$$

Thm 0.13 can be interpreted as: $u \in L^1$, $\int u = a$ and ~~u~~ $u_\varepsilon(x) = \varepsilon^{-n} \, u\left(\frac{x}{\varepsilon}\right)$, then $u_\varepsilon \to a\delta$ as distributions when $\varepsilon \to 0$

If u is a distribution, we say that $u = 0$ on the open set ~~a~~ Ω if $\langle u, \phi \rangle = 0 \;\; \forall \, \phi \in C_0^\infty(\Omega)$

The support of u is the complement of the largest open set on which $u = 0$.

Two distributions u and ~~v~~ v are said to agree on the open set Ω if $u - v = 0$ on Ω

Defining operators on distributions: T operator mapping C_0^∞ cont into itself.

Suppose: $\int (T \cancel{\phi}) \, \psi \, \cancel{\phi} = \int \phi (T' \psi) \quad \forall \, \phi, \psi \in C_0^\infty$

Then call T' the dual or transpose of T.

Extend T to act on distributions:

$\langle T u, \phi \rangle = \langle u, T' \phi \rangle$

The linear functional Tu ~~φ~~ defined in this way is continuous on C_0^∞ since T' is assumed cont.

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Examples

1) T = multiplication by C^∞ func^n f

then $T = T'$ so we can multiply a distribution u by $f \in C^\infty$: $\langle fu, \phi \rangle = \langle u, f\phi \rangle$

2) $T = \partial^\alpha$. By IBP. $T' = (-1)^{|\alpha|} \partial^\alpha$

$\therefore$ we can differentiate any distribution as often as we please to obtain other distributions

$$\langle \partial^\alpha u, \phi \rangle = (-1)^{|\alpha|} \langle u, \partial^\alpha \phi \rangle$$

3) Combine ① and ②

$T = \sum_{|\alpha| \le k} a_\alpha \partial^\alpha$ be a differential operator of order k with coeff a_α.

Integration by parts shows that the dual operator T' is given by $T'\phi = \sum_{|\alpha| \le k} (-1)^{|\alpha|} \partial^\alpha (a_\alpha \phi)$

For any distribution u:

$$\langle Tu, \phi \rangle = \langle u, T'\phi \rangle$$

Ch 3. p 80 Melrose

6.8 A parametrix for a const coeff diff operator $P(D)$ is ~~a constant~~ a coefficient differential operator if $P(D)E = \delta$ a distribution $F \in S'(\mathbb{R}^n)$ such that

(6.20) $P(D)F = \delta + \psi \qquad \psi \in C^\infty(\mathbb{R}^n)$

An operator $P(D)$ is said to be hypoelliptic if it has a parametrix satisfying

$$\text{sing supp}(F) \subset \{0\} \qquad (6.21)$$

where for any $u \in S'(\mathbb{R}^n)$

6.22 $(\text{sing supp}(u))^C = \{\bar{x} \in \mathbb{R}^n ; \exists\, \varphi \in C_c^\infty(\mathbb{R}^n),$
$\varphi(\bar{x}) \neq 0, \varphi u \in C_c^\infty(\mathbb{R}^n)\}$

C complement

Singular support of a gen. func^n u is the complement of largest open set on which u is smooth.

set sing supp (u) is closed. (since same φ works for nearby points)

Furthermore sing supp (u) $\subset$ supp(u)

In particular sing supp (u) $= \phi \rightarrow u \in S'(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$

Thm 6.9 If $P(D)$ is hypoelliptic ~~then~~

(6.25) $\text{sing supp}(u) = \text{sing supp}(P(D)u) \quad \forall u \in S'(\mathbb{R}^n)$

Lemma: 6.10 If $u \in S'(\mathbb{R}^n)$ then for any ~~diff operator~~ polynomial

Proof $\text{sing supp}(P(D)u) \subset \text{sing supp}(u) \quad \forall u \in S'(\mathbb{R}^n)$?

To prove: $\bar{x} \notin \text{sing supp}(u) \Longrightarrow \bar{x} \notin \text{sing supp}(P(D)u)$

If $\bar{x} \notin \text{sing supp}(u)$ we can find $\varphi \in C_c^\infty(\mathbb{R}^n)$, $\varphi \equiv 1$ near $\bar{x}$ such that $\varphi u \in C_c^\infty(\mathbb{R}^n)$. Then

$$P(D)u = P(D)(\varphi u + (1-\varphi)u)$$

$$= \underbrace{P(D)(\varphi u)}_{C^\infty} + P(D)((1-\varphi)u)$$

But $(1-\varphi)u = 0$ near $\bar{x}$ $\quad \varphi \equiv 1$

$\therefore \bar{x} \notin \text{supp}(P(D)(1-\varphi)u) \rightarrow \bar{x} \notin \text{sing supp}$

holds for and $P(D)$

so $\bar{x} \notin \text{sing supp}(P(D)u)$

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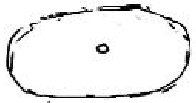
Remains to show converse:

$$\operatorname{sing\,supp}(u) \subset \operatorname{sing\,supp}(P(D)u)$$

where $P(D)$ is assumed to be hypoelliptic.

Take F, a parametrix for $P(D)$ with $\operatorname{sing\,supp} u \subset \{0\}$ and assume, or rather arrange, that F have compact support. In fact if $\bar{x} \notin \operatorname{sing\,supp}(P(D)u)$ we can arrange that

$$(\operatorname{supp}(F) + \bar{x}) \cap \operatorname{sing\,supp}(P(D)u) = \phi$$



supp F

Now $P(D)F = \delta + \varphi$

Ex 6.1. If u is holomorphic on $\mathbb{R}^n$, $\bar{\partial} u = 0$, then $u \in C^\infty(\mathbb{R}^n)$

$P(D)$ is just ~~the~~ characteristic polynomial:

$$P(\xi) = \sum_{|\alpha| \le m} c_\alpha \xi^\alpha$$

$$\widehat{P(D)u}(\xi) = P(\xi) \cdot \hat{u}(\xi) \qquad \forall\, u \in S'(\mathbb{R}^n)$$

This shows ~~that~~ we can remove $P(\xi)$ from $P(D)$ thought of as an operator on $S'(\mathbb{R}^n)$

Invert $P(D)$ by dividing by $P(\xi)$. Works well provided $P(\xi) \neq 0 \quad \forall\, \xi \in \mathbb{R}^n$.

Even the Laplacian $\Delta = \sum_{j=1}^{n} D_j^2$ doesn't satisfy this (0 at $\xi = (0, 0 \ldots 0)$)

$\therefore$ Top order derivatives to be considered:

$$p_m(\xi) = \sum_{|\alpha| = m} c_\alpha \xi^\alpha$$

is the principal symbol of $P(D)$

Def 6.11. A polynomial $P(\xi)$ or $P(D)$ is called to be elliptic of order m provided
$$P_m(\xi) \neq 0 \quad \forall\ \xi \neq 0 \in \mathbb{R}^n$$

Thm 6.12 Every elliptic differential operator $P(D)$ is hypoelliptic

We want to find a parametrix for $P(D)$, we already know ~~that~~ we might as well suppose ~~that~~ F has compact support
Taking Fourier transform of 6.27 we see ~~that~~ $\hat{F}$ should satisfy

(6.28) $$P(\xi)\hat{F}(\xi) = 1 + \hat{\varphi}, \quad \hat{\varphi} \in S(\mathbb{R}^n)$$

Here we use the fact ~~that~~ $\varphi \in C_c^\infty(\mathbb{R}^n) \subset S(\mathbb{R}^n)$ so $\hat{\varphi} \in S(\mathbb{R}^n)$ too.

First suppose ~~that~~ $P(\xi) = P_m(\xi)$ is actually homogeneous of degree m.

Thus $P_m(\xi) = |\xi|^m P_m(\hat{\xi})$, $\hat{\xi} = \dfrac{\xi}{|\xi|}$ $\xi \neq 0$.

The assumption of ellipticity means ~~that~~

(6.29) $P_m(\hat{\xi}) \neq 0 \quad \forall\ \hat{\xi} \in S^{n-1} = \{\xi \in \mathbb{R}^n, |\xi| = 1\}$

Since S^{n-1} is compact and P_m is continuous
$$|P_m(\hat{\xi})| \geq C > 0 \quad \forall\ \hat{\xi} \in S^{n-1}$$
for some constant C.

Using ~~Homogenity~~ $|P_m(\xi)| \geq C|\xi|^m$, $C > 0$ $\forall \xi \in \mathbb{R}^n$

Now to get $\hat{F}$ from 6.28 we want to divide by $P_m(\xi)$ or multiply by $1/P_m(\xi)$. The only problem is at $\xi = 0$. Avoid this by choosing $\varphi \in C_c^\infty(\mathbb{R}^n)$ as before ~~as~~ with $\varphi(\xi) = 1$ in $|\xi| \leq 1$

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Lemma 6.13 If $P_m(\xi)$ is homogeneous of degree m and elliptic ~~then~~

$$(6.32)\quad Q(\xi) = \frac{1-\varphi(\xi)}{P_m(\xi)} \in S'(\mathbb{R}^n)$$

is the Fourier Transform of a parametrix F for $P_m(D)$ satisfying (6.31) $\rightarrow P(D)F - \delta \in C^\infty(\mathbb{R}^n)$ and sing supp$(F) \subset \{0\}$

Proof: Clearly $Q(\xi)$ is a continuous function and $|Q(\xi)| \leq C(1+|\xi|)^{-m}$ $\forall\, \xi \in \mathbb{R}^n$ so $Q \in S'(\mathbb{R}^n)$. It is therefore the fourier transform of some $F \in S'(\mathbb{R}^n)$! (Rauchi crash)

Furthermore

$$\widehat{P_m(D)F}(\xi) = P_m(\xi)\hat{F} = P_m(\xi)Q(\xi) = 1-\varphi(\xi)$$

$$P_m(D)F = \delta + \psi \quad \xleftarrow{\text{Inv. Fourier}} \quad \xrightarrow{\text{with}} \quad (\hat{\psi}(\xi) = -\varphi(\xi))$$

since $\varphi \in C_c^\infty(\mathbb{R}^n) \subset S(\mathbb{R}^n)$, $\psi \in S(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$

$\boxed{\text{Thus } F \text{ is a parametrix for } P_m(D)}$

"Hard part" left: (6.33) sing supp$(F) \subset \{0\}$

consider distributions $x^\alpha F$. The idea is that for $|\alpha|$ large, x^α vanishes rather rapidly at the origin and this should weaken the singularity of F there.

Infact we shall show:

$$(6.34)\quad x^\alpha F \in H^{|\alpha|+m-n-1}(\mathbb{R}^n),\ |\alpha| > n+1-m$$

sobolev spaces are defined in terms of F.T.

show that $\widehat{x^\alpha F} \in \langle\xi\rangle^{-|\alpha|-m+n+1} L^2(\mathbb{R}^n)$

Now $\widehat{x^\alpha F} = (-1)^{|\alpha|} D_\xi^\alpha \hat{F}$ so we only need to consider behaviour of derivatives of $\hat{F}$ which is just Q in 6.32

Lemma 6.14 Let $P(\xi)$ be a polynomial of degree m satisfying

(6.35) $|P(\xi)| \geq c|\xi|^m$ in $|\xi| \geq \frac{1}{c}$ for some $c > 0$

then for constants C_α

(6.36) $$\left|\partial^\alpha \frac{1}{P(\xi)}\right| \leq C_\alpha |\xi|^{-m-|\alpha|} \quad \text{in } |\xi| > \frac{1}{c}$$

Proof

for $\alpha = 0$ it is just 6.35.

higher estimates that for each α^{th} polynomial of degree at most $(m-1)|\alpha|$ s.t

(6.37) $$\partial^\alpha \frac{1}{P(\xi)} = \frac{L_\alpha(\xi)}{(P(\xi))^{1+|\alpha|}}$$

once we know 6.37 get 6.36 straight.

$$\left|\partial^\alpha \frac{1}{P(\xi)}\right| \leq \frac{C'_\alpha |\xi|^{(m-1)|\alpha|}}{c^{1+|\alpha|}|\xi|^{m(1+|\alpha|)}} \leq C_\alpha |\xi|^{-m-|\alpha|}$$

Prove 6.37 by induction

True for $\alpha = 0$ suppose true for $|\alpha| \leq k$

To get same for each β with $|\beta| = k+1$ enough to diff one of ~~the~~ derivatives with $|\alpha| = k$ once.

Thus $$\partial^\beta \frac{1}{P(\xi)} = \partial_j \partial^\alpha \frac{1}{P(\xi)} = \partial_j \frac{L_\alpha(\xi)}{P(\xi)^{1+|\alpha|}}$$

$$L_\beta(\xi) = P(\xi)\,\partial_j L_\alpha(\xi) \qquad - \frac{(1+|\alpha|)L_\alpha \partial_j P(\xi)}{P(\xi)^{2+|\alpha|}}$$

$-(1+|\alpha|)L_\alpha(\xi)\partial_j P(\xi)$ is a polyn of degree at most $(m-1)|\alpha| + m - 1 = (m-1)|\beta|$ $\square$

Going back observe ~~that~~ $\phi(\xi) = \frac{1-\psi}{P_m(\xi)}$ is smooth in $|\xi| \leq \frac{1}{c}$ so 6.36 implies

$$|\partial^\alpha \phi(\xi)| \leq C_\alpha(1+|\xi|)^{-m-|\alpha|}$$

$$\to \langle\xi\rangle^\ell \partial^\alpha \phi \in L^2(\mathbb{R}^n) \text{ if } \ell - m - |\alpha| < -\frac{n}{2}$$

which holds if $\ell = |\alpha| + m - \frac{n}{2} - 1$ giving ~~5~~ (6.34)

By Sobolev's Embedding Thm:

$x^\alpha F \in C^k$ if $|\alpha| \leq n+1-m+k+\frac{n}{2}$

In particular if we choose $\mu \in C_c^\infty(\mathbb{R}^n)$ with $0 \notin \operatorname{supp}\mu$ then for every k, $\mu/|x|^{2k}$ is smooth and

$$\mu F = \frac{\mu}{|x|^{2k}} |x|^{2k} F \in C^{2l-2n} \quad , l > n$$

Thus $\mu F \in C_c^\infty(\mathbb{R}^n)$ and this is what we wanted to show, sing supp$(F) \subset \{0\}$.

General case:

Proof We need to show that if $P(\xi)$ is elliptic then $P(D)$ has a parametrix F as in (6.27).

Ellipticity of $P(\xi)$ implies and is equivalent to

$$|P_m(\xi)| \geq c|\xi|^m \quad , c > 0$$

On the other hand

$$P(\xi) - P_m(\xi) = \sum_{|\alpha|<m} C_\alpha \xi^\alpha$$

is a polynomial of degree at most $m-1$, so

$$|P(\xi) - P_m(\xi)| \leq c'(1+|\xi|)^{m-1}$$

if $C > 0$ is large enough then in $|\xi| > C$, $c'(1+|\xi|)^{m-1} < \frac{c}{2}|\xi|^m$

so $$|P(\xi)| \geq |P_m(\xi)| - |P(\xi) - P_m(\xi)|$$

$$\geq c|\xi|^m - c'(1+|\xi|)^{m-1} \geq \frac{c}{2}|\xi|^m$$

$\therefore P(\xi)$ satisfies conditions of Lemma 6.14.

Thus if ~~and~~ $\varphi \in C_c^\infty(\mathbb{R}^n) = 1$ in a large enough ball ~~then~~ $Q(\xi) = \frac{1-\varphi(\xi)}{P_m(\xi)}$ is in C^∞ and satisfies 6.36 which can be written as:

$$|D^\alpha Q(\xi)| \leq C_\alpha (1+|\xi|)^{m-|\alpha|}$$

Defining $F \in S'(\mathbb{R}^n)$ by $\hat{F}(\xi) = g(\xi)$ gives soln to 6.27.

Last step: if $F \in S'(\mathbb{R}^n)$ has compact supp and satisfies 6.27 then

$u \in S(\mathbb{R}^n)$, $P(D)u \in S'(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$

$u = F * (P(D)u) - \psi * u \in C^\infty(\mathbb{R}^n)$

Define
Prop 6.15: If $f \in S'(\mathbb{R}^n)$ and $\mu \in S'(\mathbb{R}^n)$ has compact supp ~~then~~

$$\text{sing supp}(u * f) \subset \text{sing supp}(u) + \text{sing supp}(f)$$

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i) support of func^n f is the set :

$\text{supp} f = \overline{\{ x \in \mathbb{R} : f(x) \neq 0 \}}$ (closure)

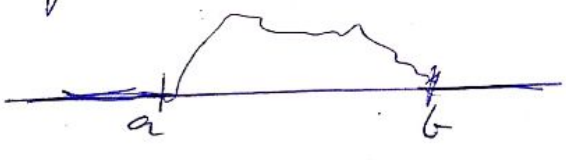
~~Anything~~ Anywhere outside support function vanishes.

ii) f has compact support if ~~f~~ supp f is bounded.

(Heine - Borel: Bounded and closed $\Leftrightarrow$ compact in $\mathbb{R}^n$)

i.e. for $f : \mathbb{R} \to \mathbb{R}$

if $\exists$ $a, b \in \mathbb{R}$ s.t. ~~compact~~ support $f \subseteq [a, b]$



iii) $C_c(\mathbb{R}) :=$ space of all cont. func^{n}s with compact support.

iv) $C_0(\mathbb{R}) :=$ $f : \mathbb{R} \to \mathbb{R}$ s.t. f is continuous and $f(x) \to 0$ as $x \to \pm\infty$

$C_R(\mathbb{R}) := f$:

$$n(x) := \begin{cases} C \exp\left(\frac{1}{|x|^2 - 1}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

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Def $f: U \to \mathbb{R}$ is locally integrable, define its mollification:

$$f^{\varepsilon} := \eta_{\varepsilon} * f \quad \text{in } U_{\varepsilon}$$

ie: $$f^{\varepsilon}(x) = \int_{U} \eta_{\varepsilon}(x-y) f(y) dy$$

$$= \int_{B(0,\varepsilon)} \eta_{\varepsilon}(y) f(x-y) dy$$

for $x \in U_{\varepsilon}$

3)

4) part 2

Parseval's identity: fundamental result on summability of Fourier series of a func^n:
Geometrically: ~~Py~~ Pythagorean Theorem for inner-product spaces

$$\|f\|^2_{L^2_p(-\pi,\pi)} = \int_{-\pi}^{\pi} |f(x)|^2 dx = 2\pi \sum_{n=-\infty}^{\infty} |c_n|^2$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

Plancherel:

$$\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

Uncovering Moment space:

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int \Phi(k) e^{ikx} dk$$

$$\Phi(k) = \frac{1}{\sqrt{2\pi}} \int \psi(x) e^{-ikx} dx$$

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int \frac{1}{\sqrt{2\pi}} \int \psi(x') e^{-ikx'} dx'$$

$$= \int dx' \, \psi(x') \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, e^{ik(x-x')}}_{\delta(x'-x)}$$

$$\delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ik(x-x')}$$

$$\int dx\, \underbrace{\psi^*(x)}_{k}\, \underbrace{\psi(x)}_{k'} =$$

$$\int dx\, \frac{1}{\sqrt{2\pi}}\int dk\, \Phi^*(k)\, e^{-ikx}\, \frac{1}{\sqrt{2\pi}}\int dk'\, \Phi(k')\, e^{ik'x}$$

$$= \int dk\, \phi^*(k) \int dk'\, \phi(k')\, \frac{1}{2\pi}\underbrace{\int dx\, e^{i(k'-k)x}}_{\delta(k'-k)}$$

$$= \int dk\, \phi^*(k)\, \phi(k)$$

$$\boxed{\int dx\, |\psi(x)|^2 = \int dk\, |\phi(k)|^2}$$

Parseval's thm

Plancherel's Thm.

1) $x_{[-1,1]}(x) = \begin{cases} 1 & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

Fourier Transform :

$$F\{x_{[-1,1]}(x)\} = \int_{-\infty}^{\infty} x_{[-1,+1]}(x)\, e^{i2\pi x\xi}\, dx$$

$$= \int_{-\infty}^{-1} (0)\, dx + \int_{-1}^{1} \text{''} \quad \text{''} + \int_{1}^{\infty} (0)\, dx$$

$$= \int_{-1}^{1} 1 \times e^{i2\pi x\xi}\, dx$$

$$= \frac{e^{2\pi i x \xi}}{2\pi i \xi}\Bigg|_{-1}^{1}$$

$$= \frac{e^{2\pi i n \xi} - e^{-2\pi i x \xi}}{2\pi i \xi}$$

$$= \frac{\cancel{2}\cancel{i}\sin(2\pi\xi)}{\cancel{2}\pi \cancel{i}\xi}$$

$$= \frac{\sin(2\pi\xi)}{\pi\xi}$$

$\xi \to \infty \quad F\{\chi\}(\xi) \leq \frac{1}{\pi\xi}$ $\quad (\because \sin(n) \leq 1)$

$\therefore$ decays as $\propto \frac{1}{\xi}$ or (ξ^{-1})

is not in the Schwartz class $\because \xi^2 F\{\chi\}(\xi) \propto \xi$ which goes unbounded as $\xi \to \infty$

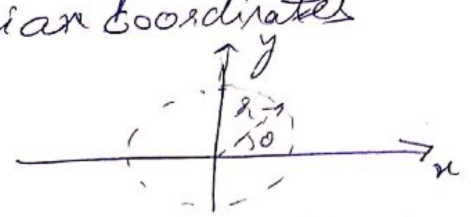
that our starting function $\chi_{[-1,1]}(x)$ is ~~/~~ not in $\mathcal{S}$ either

hence this is possible

2) Laplacian in polar coordinates for $\mathbb{R}^2$:

$\Delta = \partial_{xx} + \partial_{yy}$ in cartesian coordinates

To convert to polar: $x = r\cos\theta$
$y = r\sin\theta$



Area element: is given by change of coordinates with Jacobian determinant

$$dx\,dy = |(J)|\,dr\,d\theta$$

$$dx\,dy = \left|\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix}\right| dr\,d\theta$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} dr\,d\theta$$

$$= r(\cos^2\theta + \sin^2\theta)\,dr\,d\theta$$

$$dx\,dy = r\,dr\,d\theta = (1\,dr)\,(r\,d\theta)$$

∴ measure along $r = 1$ and along θ: r

gradient in an orthogonal coordinate system is: $\hookrightarrow (u, v)$

$$\vec{\nabla} f = \left(\frac{1}{g}\frac{\partial}{\partial u}\right)\hat{u} + \left(\frac{1}{h}\frac{\partial}{\partial v}\right)\hat{v}$$ where g, h are respective measures.

Here

$$\therefore \vec{\nabla} = \left(\frac{1}{1}\frac{\partial}{\partial r}\right)\hat{r} + \left(\frac{1}{r}\frac{\partial}{\partial \theta}\right)\hat{\theta}$$

Divergence in orthogonal coordinates is given by

$$\vec{\nabla}\cdot = \frac{1}{gh}\left(\frac{\partial(h\,)}{\partial u}\right) + \frac{1}{gh}\left(\frac{\partial g(\,)}{\partial v}\right)$$

here $$\vec{\nabla}\cdot\vec{f} = \frac{1\,\partial(r f_r)}{(1)\,\partial r} + \frac{1\,\partial(1 f_\theta)}{r\,\partial\theta}$$

substituting for f as gradient:
we get the Laplacian
$$\nabla^2 f = (\vec{\nabla} \cdot (\vec{\nabla f}))$$
Laplacian = divergence of gradient

$$\nabla^2 f = \frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right)$$

$$\nabla^2 = \boxed{\boxed{\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}}$$ Laplacian in spherical coordinates

Radial Harmonic functions satisfy:
$$\Delta f = \Delta f(r) = 0 \qquad \Delta = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2$$

$$\therefore \frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r} f(r)\right) + \frac{1}{r^2}\frac{\partial^2 f(r)}{\partial \theta^2} = 0$$

Only radial dependence

$$\frac{d}{dr}\left(r\frac{d}{dr} f(r)\right) + 0 = 0$$
Integrate wrt r
$$\therefore r\frac{d}{dr} f(r) = \text{constant} = A \quad \text{(say)}$$

$$\therefore \frac{d}{dr} f(r) = \frac{A}{r} \qquad (\text{for } r \neq 0)$$

$$\int d f(r) = \int A \frac{dr}{r}$$
Integrating wrt r
$$\boxed{f(r) = A \ln(r) + B} \quad r \neq 0,\ A, B \in \mathbb{R} \text{ constants}$$

Given $\Omega = \mathbb{R}^2 \setminus B_1(0) \rightarrow r \neq 0$ satisfied.
Boundary value: 1 on $\partial B_1(0)$
$$\rightarrow f(1) = 1$$
$$A \ln(1) + B = 1 \qquad (\ln(1) = 0)$$
$$\boxed{B = 1} \rightarrow \boxed{f(r) = A\ln(r) + 1}$$

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4) $u(t,x) = \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{(x-y)^2}{4t}} f(y)\,dy$

solves $\partial_t u = \Delta u$ on $\mathbb{R}^n$ with init: $u(0,x) = f(x)$

If $f \in C_c^\infty(\mathbb{R}^n) = X$ ~~then~~ consider ~~a closed path in~~ ~~complex~~ $\mathbb{C}^n$ ~~γ:~~ $x = z = (x_1, x_2, \dots, x_n) + i(y_1, y_2, \dots)$

$\cancel{\oint u(t,z)\,dz}$

$u(t,x) = \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{(x-y)^2}{4t}} f(y)\,dy$

for $f(y)$ compactly supported with supp $= X \subset \mathbb{R}^n$, integral simplifies to

① $u(t,x) = \int_X \frac{1}{(4\pi t)^{n/2}} e^{-\frac{(x-y)^2}{4t}} f(y)\,dy$

This gives $u(t,x)$ defined in $T \times \mathbb{R}^n$

Consider the function defined on $\mathbb{C}^n$

$u(t,z) = \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{(z-y)^2}{4t}} f(y)\,dy$

For $z =$ Real This matches exactly with ①

~~(compact support renders extension of domain for integral harmless)~~

If we can prove ~~that~~ this newly defined function is complex analytic, ~~this~~ gives us at least one possible complex analytic function which is the extension needed. Uniqueness we don't need to consider but may be established by appealing to biholomorphicity.

Now that; $z = a + ib$

$$\Rightarrow u(t, z) = \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{(a+ib-y)^2}{4t}} f(y)\,dy$$

$$= \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{[(a-y)+ib]^2}{4t}} f(y)\,dy$$

$$|u(t,z)| = \left| \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{(a^2+y^2-2ay-b^2)}{4t}} e^{-2i\frac{(a-y)b}{4t}} f(y)\,dy \right|$$

$$= \int_{X} \frac{1}{(4\pi t)^{n/2}} \left| e^{-[(a-y)^2-b^2]} \right| \cancel{\left| e^{-2i\frac{(a-y)b}{4t}} \right|} |f(y)|\,dy$$

$$= \int_X \frac{e^{-(a-y)^2+b^2}}{(4\pi t)^{n/2}} |f(y)\,dy|$$

$f(y) \in C_c^\infty(\mathbb{R}^n) \to$ bounded by $\int_X f(y)\,dy$

$$\therefore |u(t,z)| \le \frac{M}{(4\pi t)^{n/2}} \int_X e^{-[(a-y)^2-b^2]}\,dy$$

$$\le \frac{M e^{b^2}}{(4\pi t)^{n/2}} \int_X e^{-(a-y)^2}\,dy$$

which is also bounded

But $e^{b^2} \to \infty$ for $b \to \pm\infty$

$\therefore$ This function is not bounded on $\mathbb{C}^n$.

But bounded ~~for~~ on any closed path γ in $\mathbb{C}^n$

$\therefore$ Fubini's theorem can be applied as e^{b^2} is bounded for $b \in (-\infty, \infty)$

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Take a closed curve γ in $\mathbb{C}^n$

$$\oint_\gamma u(t,z)\,dz = \oint_\gamma u(t,z)\,dz$$

$$= \oint_\gamma \left(\int_X \frac{e^{-\frac{(z-y)^2}{4t}}}{(4\pi t)^{n/2}} f(y)\,dy \right) dz$$

Applying Fubini's thm

$$= \int_X \left(\oint_\gamma e^{-\frac{(z-y)^2}{4t}} f(y)\,dz \right) dy$$

$$= \int_X \oint_\gamma \left(e^{-\frac{(z-y)^2}{4t}} dz \right) f(y)\,dy$$

$=$ This function is holomorphic on $\mathbb{C}^n$ (entire)

$\therefore \oint_\gamma u(t,z)\,dz = 0$ for any closed path γ in $\mathbb{C}^n$

$\xrightarrow{\text{Morera's thm}}$ $u(t,z)$ is complex analytic and the required complex extension.

(*) Completely missed reading " and $t \in \mathbb{C}$ such that $\operatorname{Re} t > 0$ "

Redo : modify the func by : defining

$\tilde{t} = c + id$

$$u(\tilde{t}, z) = \int_{\mathbb{R}^n} \frac{1}{(4\pi(c+id))^{n/2}} e^{-\frac{(a+ib-y)^2}{4(c+id)}} f(y)\,dy$$

$$= \int_X \frac{1}{4\pi(c+id)^{n/2}} e^{-\frac{(a+ib-y)^2}{4(c+id)}} f(y)\,dy$$

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$$e^{-\frac{(a+ib-y)^2}{4(c+id)}} = e^{-\frac{((a-y)^2-b^2+2(a-y)ib)(c-id)}{4(c+id)(c-id)}}$$

$$= e^{-\frac{((a-y)^2-b^2)c+2(a-y)bd+i[2b(a-y)c-d((a-y)^2-b^2)]}{4(c^2-d^2)}}$$

$$= e^{-\frac{((a-y)^2-b^2)c+2(a-y)bd}{4(c^2-d^2)}} \, e^{-i\frac{(2b(a-y)c-d((a-y)^2-b^2))}{4(c^2-d^2)}}$$

$\downarrow$ bounded for any path γ in $\mathbb{C}^n$

~~Re~~ ~~Re~~

$\frac{1}{4\pi(c+id)^{n/2}} \rightarrow$ if $c=d=0$ blows up

but for $Re(t)>0$

$\longrightarrow c>0 \quad \therefore$ Not a problem.

$\frac{\exp p(z)}{q(z)} \rightarrow$ is holomorphic for $q(z)\neq 0$

$p(z), q(z) \rightarrow$ polynomials.

$\therefore$ Applying previous result on $\oint_\gamma$ is justified.

ok