Solitary Waves and Solitons

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15D110001

Nonlinear Dynamics (PH 542)

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Intro

Millenium Prize Problem (Yang Mills and Mass Gap)

https://www.youtube.com/watch?v=Ederft9dkag

Solitons

QFT

String Theory (Polyakov Action)

Oscillators 1

- Linear Harmonic
- Parametric
- Continuously Modulated
- Pulse
- Step Modulated

Oscillators 2

- Simple Pendulum
- Nonlinear Lienard
- Duffing
- Van der Pol

History

“ I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of waterin the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed.

I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel.

Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.”

-John Scott Russel

Soliton as a self sufficient dynamic entity

Used to model dynamic behaviour of systems in

Hydrodynamics to nonlinear optics,

from plasmas to shock waves,

from tornados to the Great Red Spot of Jupiter,

from the elementary particles of matter to the elementary particles of thought

Korteweg-De Vries Equation (KdV)

Non-linear Partial Differential Equation (PDE) of third order.

u t + 6 u u x + u x x x = 0

Auto-Backlund Transformation

Cauchy Riemann conditions

$$u_x = v_y, \qquad u_y = -v_x$$

$$u_{xx} + u_{yy} = 0, \qquad v_{xx} + v_{yy} = 0$$

KdV Auto-Backlund Transform

Substitute $$u = z_x$$

$$z_{tx} + 6z_x z_{xx} + z_{xxxx} = \partial_x[z_t + 3z_x^2 + z_{xxx}]$$

Integrate $$z_t + 3z_x^2 + z_{xxx} = f(t)$$

KdV Auto-Backlund Transform

Introducing a new variable w by shifting the v

$$w = z - \int^{t} f(t')dt'$$

without loss of generality

$$w_t + 3w_x^2 + w_{xxx} = 0$$

KdV Auto-Backlund Transform

- Auto-Backlund transformation for the above equation is the following

$$v_x + w_x = \beta - \frac{1}{2}(v - w)^2$$

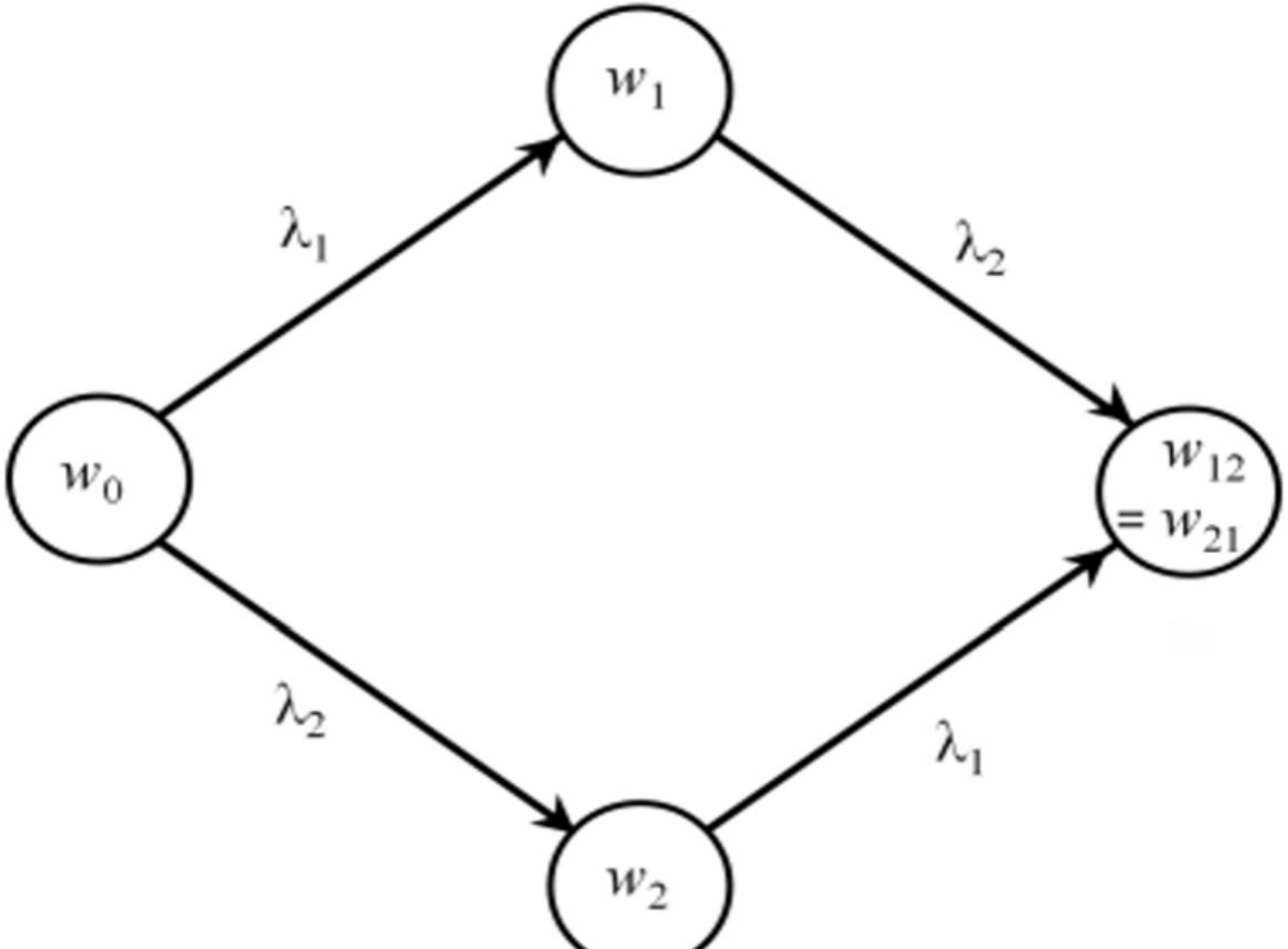
$$v_t + w_t = (v - w)(v_{xx} - w_{xx}) - 2(v_x^2 + v_x w_x + w_x^2)$$

Solutions

$$w = \sqrt{2\beta}\tanh[\sqrt{\frac{\beta}{2}}x + \alpha(t)]$$

$$\bar{w} = \sqrt{2\beta}\coth[\sqrt{\frac{\beta}{2}}x + \alpha(t)]$$

Bianchi’s Theorem of Permutability



[Dra-92, p115]

Non-Linear Superposition

$$w_{1x}(\beta_1) = -w_x + \beta_1 - \frac{1}{2}(w_1(\beta_1) - w)^2$$

$$w_{2x}(\beta_2) = -w_x + \beta_2 - \frac{1}{2}(w_2(\beta_2) - w)^2$$

$$w_{12x}(\beta_1, \beta_2) = -w_{2x}(\beta_2) + \beta_1 - \frac{1}{2}(w_{12}(\beta_1, \beta_2) - w_2(\beta_2))^2$$

$$w_{21x}(\beta_2, \beta_1) = -w_{1x}(\beta_1) + \beta_2 - \frac{1}{2}(w_{21}(\beta_2, \beta_1) - w_1(\beta_1))^2$$

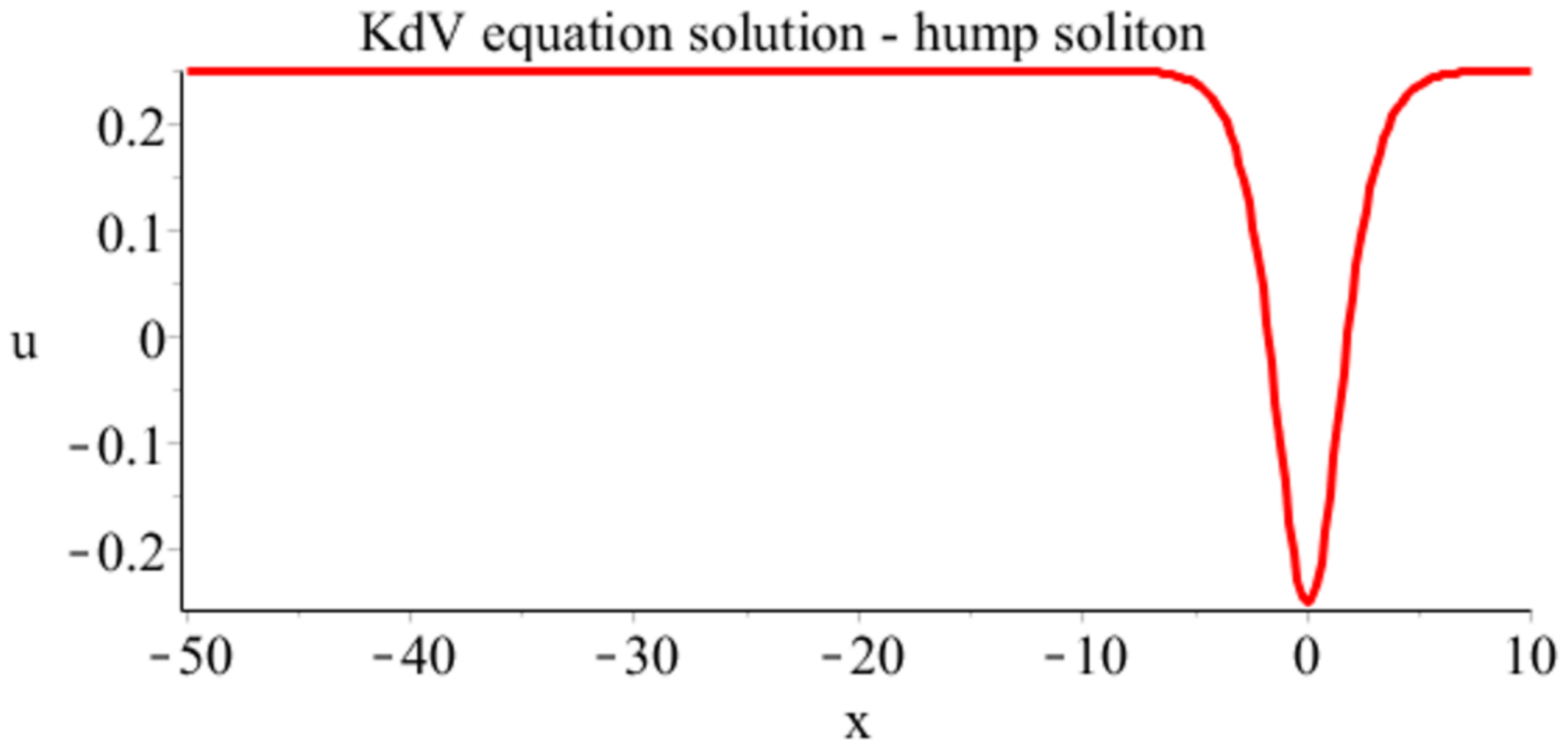
Multi Soliton Solution

$$W = w + \frac{2(\beta_1 - \beta_2)}{[w_1(\beta_1) - w_2(\beta_2)]}$$

$$W = \frac{2(\beta_1 - \beta_2)}{[w(\beta_1) - \bar{w}(\beta_2)]}$$

Two Soliton Solution

$$y[x,t] = \frac{-\sqrt{2}(b_1 - b_2)\left[b_1 \operatorname{sech}^2\left(\sqrt{\frac{b_1}{2}}\left(x - 2b_1 t\right)\right) + b_2 \operatorname{csch}^2\left(\sqrt{\frac{b_2}{2}}\left(x - 2b_2 t\right)\right)\right]}{\left[\sqrt{b_1}\tanh\left(\sqrt{\frac{b_1}{2}}\left(x - 2b_1 t\right)\right) - \sqrt{b_2}\coth\left(\sqrt{\frac{b_2}{2}}\left(x - 2b_2 t\right)\right)\right]}$$



KdV equation solution - hump soliton
u
0.2
0.1
0
-0.1
-0.2
-50
-40
-30
-20
-10
0
10
x

Two Soliton collision

“Kink collision” preserves form

https://www.youtube.com/watch?v=bleWLaECCkM

Hirota’s Bilinear Method for Soliton Equations

- Rewrite KdV eqn as Log of Tau function (Painleve)
- No algorithm for transforming to bilinear form
- Lax Pair for KP

Yang Baxter Equation

- Represents sufficient condition for quantum integrability.
- Lie Symmetries Analysis (similarity method).

Some Applications

- Falaco Solitons and Topological Invariants
- Inverse Scattering Method
- Lax Pairs
- Nonlinear Diffusion
- Nonlinear Klein Gordon Equations
- Nonlinear Schrodinger Equation
- Maxwell Bloch Equations
- Toda Chain
- Quantum Field Theory, Lattice Gauge Theory
- Magnetic Monopoles, t'Hooft-Polyakov monopole
- Euclidean Yang-Mills Instantons

References

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- P. G. Drazin: Solitons
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