Lecture Notes on NLD — Index Theory

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3.8. Index theory

Consider a smooth 2D vector field $V(\varphi)$. The angle ~~that~~ vector V makes with respect to $\hat{\varphi}_1$ and $\hat{\varphi}_2$ axes is a scalar field:

$$\theta(\varphi) = \tan^{-1}\left(\frac{V_2(\varphi)}{V_1(\varphi)}\right)$$

As long as V has finite length [doesn't vanish or blow up] the angle θ is well defined. Expect ~~that~~ we can integrate $\nabla\theta$ over a closed curve C in phase space to get

$$\oint_C d\varphi \cdot \nabla\theta = 0$$

[θ at all points is independent of path taken]

[$\nabla\theta$ is gradient of a scalar field]

However ~~this~~ can fail if $V(\varphi)$ vanishes or diverges at one or more pts. in the interior of C.

Define:
$$W_C(V) = \frac{1}{2\pi}\oint_C d\varphi \cdot \nabla\theta$$

then $W_C(V) \in \mathbb{Z}$ is an integer valued function of C which ~~is the~~ change in θ around the curve C. This must be an integer because θ is well defined only upto multiples of 2π. Differential changes of θ are in general well defined.

If $V(\varphi)$ is finite → neither diverges nor vanishes anywhere in int(C) then $W_C(V) = 0$.

Assuming ~~that~~ V never diverges, any singularities in θ must be arising from pts where $V = 0$, which in general occurs at isolated points. since two variables

The index of a two-dimensional vector field $V(\varphi)$ at a point φ is the integer-valued winding of V about that point:

$$\underset{\varphi_0}{\mathrm{ind}}(V) = \lim_{a \to 0} \frac{1}{2\pi} \oint_{C_a(\varphi_0)} d\varphi \cdot \nabla \theta$$

$$= \lim_{a \to 0} \frac{1}{2\pi} \oint d\varphi \cdot \left(\frac{V_1 \nabla V_2 - V_2 \nabla V_1}{V_1^2 + V_2^2} \right)$$

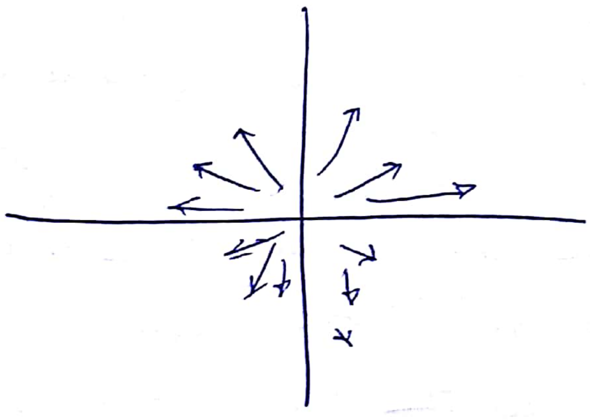
$$\left[\because \theta = \tan^{-1}\left(\frac{V_2}{V_1}\right) \right.$$

$$\left. d\theta = \frac{1}{1+\left(\frac{V_2}{V_1}\right)^2} \times \left(\frac{V_1 dV_2 - V_2 dV_1}{V_1^2} \right) \right]$$

The index of a closed curve C is given by the sum of the indices at all the singularities enclosed by the curve*:

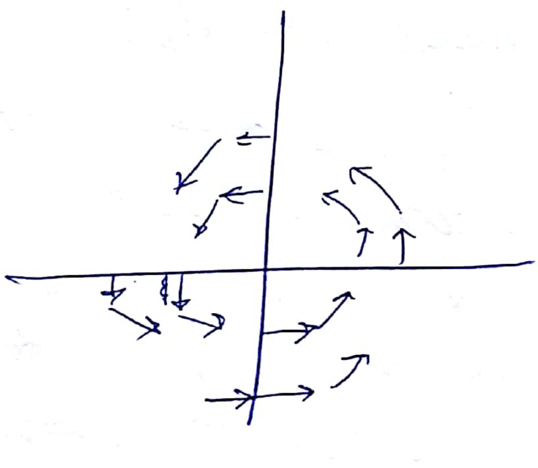
$$W_C(V) = \sum_{\varphi_i \in \mathrm{int}(C)} \underset{\varphi_i}{\mathrm{ind}}(V)$$

* Technically should weight the index of each enclosed singularity by the signed number of times the curve C encloses that singularity. For simplicity and clarity, assume that curve C is homeomorphic to the circle S^1.



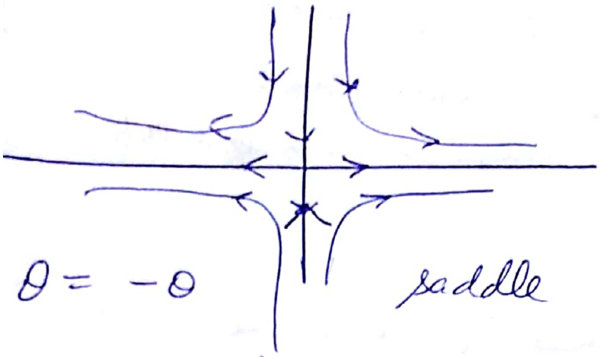
Index +1

Unstable Node.

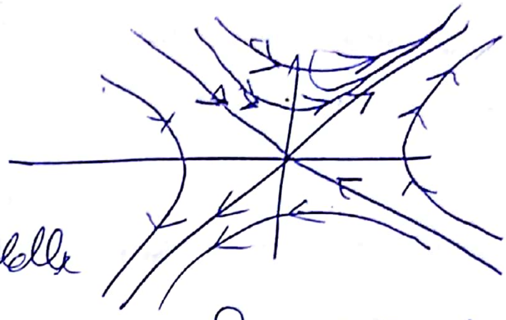


+1

Periodic cycle (Purely imaginary)



$\theta = -\theta$ saddle



saddle

$\theta = \frac{\pi}{2} - \theta$

Index -1 $\qquad$ -1

1st diagram $v = (x, y) \to \theta = \theta$ $\qquad$ $v = (-y, x)$ $\to \theta = \theta + \frac{\pi}{2}$

$\boxed{+1}$

2nd $v = (x, -y)$ $\to \theta = -\theta$

$v = (y, x)$ $\to \theta = -\theta + \frac{\pi}{2}$ $\qquad \boxed{-1}$

$v = (x^2 - y^2, 2xy)$

$\theta = 2\theta$

$+2$

$\to$ single fixed pt at $(0,0)$ index $+2$

$v = (1 + x^2 - y^2, x + 2xy)$

$\downarrow$

two fixed pts: $(x^*, y^*) = \begin{cases} (0, 1) \\ (0, -1) \end{cases}$

$J = \begin{pmatrix} 2x & -2y \\ 1+2y & 2x \end{pmatrix}$

$M_{0,1} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$ $\quad M_{0,-1} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$

each $+1$ $\quad +1$

$\therefore W_C(V) = +1 + (+1) = +2$

<u>Properties</u> of index/winding no.

- The index $I_{\varphi_0}(V)$ of $N = 2$ vector field V at pt φ_0 is winding no. of V about ~~that~~ point.
- The winding number $W_C(V)$ of a curve C is the sum of indices of the singularities enclosed by ~~that~~ curve.
- Smooth deformations of C do not change the winding no. One must stretch C over a fixed pt singularity in order to change $W_C(V)$.
- Uniformly rotating each vector in the vector field by angle β has the effect of sending $\theta \to \theta + \beta$. This leaves all indices and winding no.s invariant.

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- Nodes and spirals whether stable or unstable have index of +1 (so do the special cases of centers, stars and degenerate nodes)
- saddle points have index −1.
- Clearly any closed orbit must lie on a curve C of index +1.

3.6.1 Gauss–Bonnet Thm

Deep result in mathematics: connects local geometry of 2D manifold to global topological structure.

Content of theorem:

$$\int_M dA\, K = 2\pi\, \chi(M) = 2\pi \sum_i \mathrm{ind}_{\varphi_i}(V)$$

where M is a 2D manifold (a topological space locally homeomorphic to $\mathbb{R}^2$)

K is the local gaussian curvature of M:

$K = (R_1 R_2)^{-1}$ where R_1, R_2 are principal radii of curvature.

$\chi(M)$: Euler characteristic of M

given by $\chi(M) = 2 - 2g$ where g is the genus of M, which is the no. of holes (or handles) of M.

$V(\varphi)$ is any smooth vector field on M, φ_i are the singularity points of the vector field, which are fixed pts of the dynamics $\dot{\varphi} = V(\varphi)$

Consider $M = S^2$ unit 2-sphere.

at any pt $R_1 = R_2 = 1 \to K = 1$ $\int_M dA\, K = 4\pi$

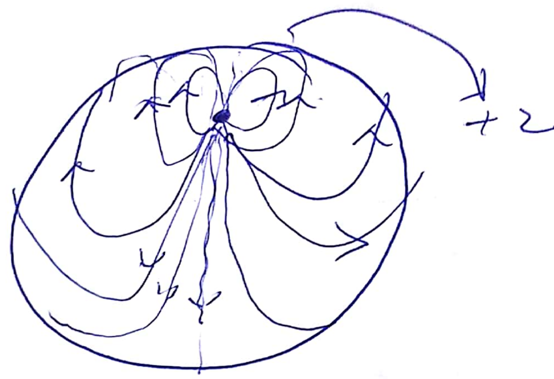
$4\pi = (2\pi)\, \chi(M) \to \boxed{\chi(M) = 2} = 2 - 2g = 2$

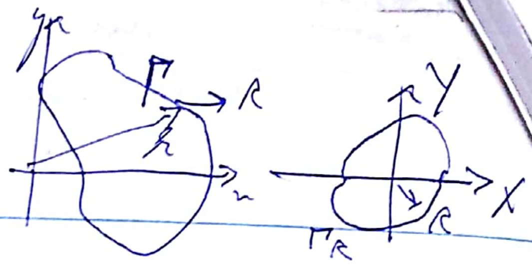
$\to g = 0$ for 2-sphere

Also any smooth vector field on S^2 must have a singularity or singularities

$\sum \mathrm{ind}(V) = \chi(S^2) = 2$


(+1)
(+1)


+2



$X = \dot{x} \qquad Y = \dot{y}$

$$\frac{d\phi}{ds} = \frac{X\dot{Y} - Y\dot{X}}{X^2+Y^2}$$

$$I_\Gamma = \frac{1}{2\pi}\oint_s \frac{d\phi}{ds}\, ds = \frac{1}{2\pi}\oint_s \frac{X\dot{Y} - Y\dot{X}}{X^2+Y^2}\, ds$$

$$\boxed{I_\Gamma = \frac{1}{2\pi}\oint \frac{X\,dY - Y\,dX}{X^2+Y^2}}$$

$r(s) = (x(s), y(s))$ describes Γ

$R(s) = (X, Y)$ can be regarded as a position vector on a plane with X, Y axis describes another curve Γ_R.

Γ_R is closed since R returns to its original value after a complete cycle

Γ_R encircles origin I_Γ times, anticlockwise

Thm 3.1 Suppose on and inside Γ: X, Y and their 1st derivatives are cont and X and Y are not simultaneously 0. (No eqlbm pts)
Then I_Γ is 0.

Green's Thm

$$I_\Gamma = \frac{1}{2\pi}\iint_{S_R} \left[\frac{\partial}{\partial X}\left(\frac{X}{X^2+Y^2}\right) + \frac{\partial}{\partial Y}\left(\frac{Y}{X^2+Y^2}\right)\right] dX\,dY$$

S_R ↳ region enclosed by Γ_R

Integrand is identically 0. ($u_x = -v_y$

Green's Theorem

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

8 Index Theory — Strogatz NLD Lec 8 Youtube

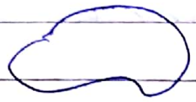
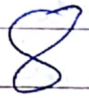
~~Aim~~ Provides "global" info about phase portrait.

Linearizing: local methods → However depends on info in infinitesimal neighbhd of pt

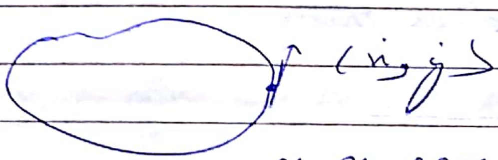
Topology: Complex - winding No. Physics: Gauss.

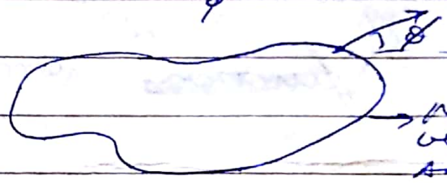
Index of a closed curve C → (doesn't intersect itself & and doesn't pass through fixed pt.)

C = simple closed curve, not necessarily a closed trajectory

  → not simple.

$\phi = \tan^{-1}[\dot{y}/\dot{x}]$





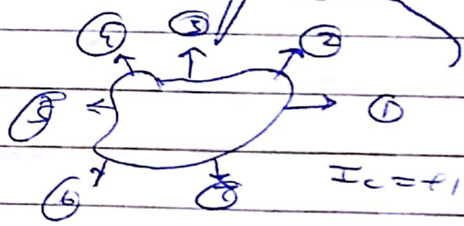
→ need not be trajectory any closed curve

as x goes around C once, counterclockwise, ϕ changes continuously.

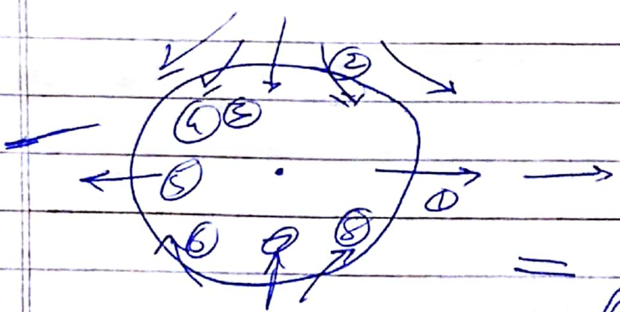
If $\dot{x}$ $\dot{y}$ are continuous, Let $[\phi]_C$ = net change in ϕ around C. Then $I_C = \frac{1}{2\pi}[\phi]_C$ is index of C wrt vector field $(\dot{x}, \dot{y})$

$[\phi]_C = 2\pi$

$I_C = +1$



$I_C = +1$



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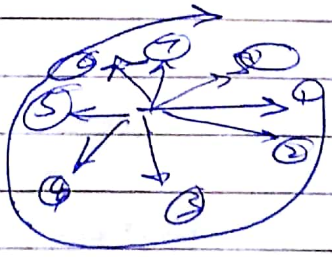
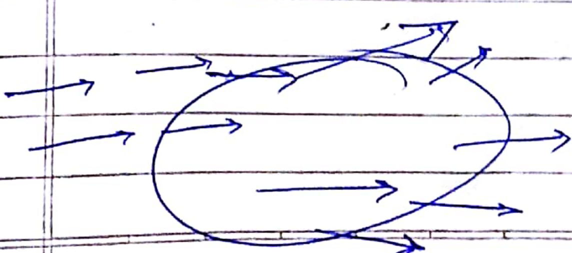


$I_C = -1$



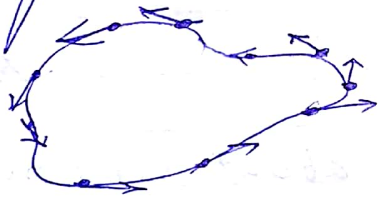
$I_C = 0$

... stable or unstable

//

Properties :

1) Index of a closed trajectories



Any closed trajectory has index

$\boxed{I_c = +1}$

Questions ?

Berry Phase curvature and relⁿ to index theory

Brouwer's Fixed pt thm and sperner's lemma } Topology

Triangulation and Index theory

green's function, divergence, Stokes' thm

Properties of the Index

1) Suppose C can be deformed continuously into C' without passing through a fixed pt. Then $I_C = I_{C'}$

(Deforming C into C' continuous then $[\phi]_C / 2\pi = \oint d\phi / 2\pi$ changes continuously. But I_C is an integer. Hence it can't change without jumping

2) If C doesn't enclose any fixed points, then $I_C = 0$

By property ①, shrink C to a tiny circle without changing index. But ϕ is essentially constant on such a circle, because all vectors point in nearly same direction thanks to the assumed smoothness of the vector field. Hence $[\phi]_C = 0$ $I_C = 0$

3. If we reverse all arrows in vector field by changing $t \to -t$, the index is unchanged.

$\dot{x} = \frac{dx}{dt} = f(x)$ Let $dt = -dt'$ $\to \dot{x}_{\text{in } t'} = -f(x)$

All angles change from ϕ to $\pi + \phi$ $[\phi]_C$ stays

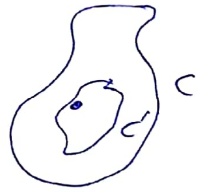
4. Suppose closed curve C is actually a trajectory for system, i.e. C is a closed orbit. Then $I_C = +1$

vector field is everywhere tangent to C because C is a trajectory

Index of a pt.

Let x^* be an isolated fixed pt.

I_C is independent of C hence only dependent on x^*.



⊛ Index not related to stability

Spirals, centers, degenerate nodes and stars all have $I = +1$

⊛ saddle pt. is truly different.

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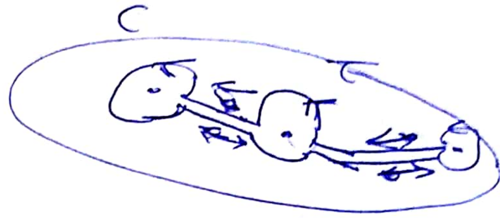
Thm : If a closed curve C surrounds n isolated fixed pts. $x_1^*, x_2^*, \ldots x_n^*$. then



$$I_C = I_1 + I_2 + \ldots + I_n$$

I_k is index of x_k^* for $k = 1, \ldots n$

Thm Any closed orbit in the phase plane must enclose fixed pts whose indices sum to $+1$.

Corollary: Always at least one fixed pt inside any closed orbit in the phase plane.

If there is only one fixed point inside it, it cannot be a saddle point.

// Differential Eqn.

In every case the change in ϕ must be a multiple of 2π.

$$[\phi]_{\Gamma} = 2\pi I_{\Gamma} \qquad 3.4$$

where I_Γ is defined and is an integer.
Index of Γ wrt f vector field (X, Y)
(Γ is always counterclockwise)

Algebraic representn of I_Γ:

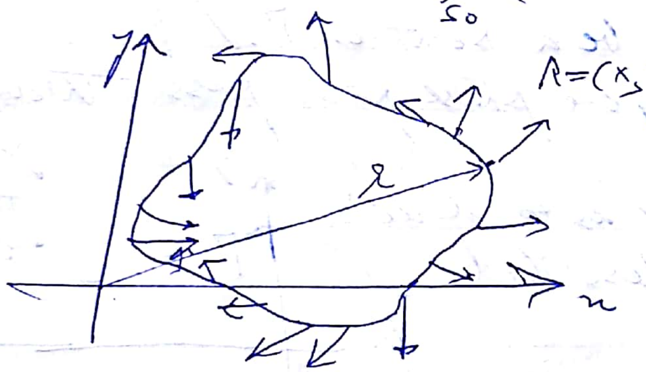
$r(s) = (x(s), y(s)) \quad s_0 < s < s_1$

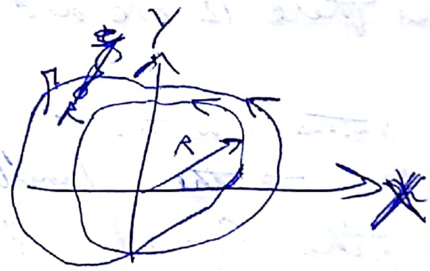
$$\frac{d \tan\phi(s)}{ds} = \frac{d}{ds}\left(\frac{Y}{X}\right)$$

$$\therefore \frac{d\phi}{ds} = \frac{XY' - YX'}{X^2+Y^2}$$

from 3.4

$$I_\Gamma = \frac{1}{2\pi}\int_{s_0}^{s_1}\left(\frac{d\phi}{ds}\right)ds = \frac{1}{2\pi}\int_{s_0}^{s_1}\frac{XY' - YX'}{X^2+Y^2}\,ds$$





As $r(s)$ describes Γ, $R(s) = (X, Y)$ describes another curve Γ_R on XY axes plane.
Γ_R is closed since it returns to original value after complete cycle.

Γ_R encircles origin I_Γ times.

$$I_\Gamma = \frac{1}{2\pi}\oint_{\Gamma_R}\frac{X\,dY - Y\,dX}{X^2+Y^2}$$

$\left(\cdots\frac{dY}{dt} - \cdots\frac{dX}{dt}\right)dt$

$\left(\cdots dY - \cdots dX\right) \times \left(\frac{dt}{dt}\right)$

take parameter $s = t$, $ds = dt$

Thm Suppose Γ lies in a simply connected region on which X, Y and ~~their~~ 1st derivatives are continuous and X and Y are not simultaneously 0 (No equi pts)

Then I_Γ is 0

Proof: Green's thm $\oint P\,dx + Q\,dy = \iint_{D_\Gamma}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$

We can write $\frac{dX}{ds} = X_x\frac{dx}{ds} + X_y\frac{dy}{ds}$ same for Y then $I_\Gamma = \frac{1}{2\pi}\oint_\Gamma\left(\frac{XY_x - YX_x}{X^2+Y^2}dx + \frac{XY_y - YX_y}{X^2+Y^2}dy\right)$

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Date

Chapter 3 Geometrical Aspects of Plane Autonomous systems

— Jordan, Smith

Intro where the linear approximation is 0, index of a point provides supporting Inform^n on its nature and complexity in strongly nonlinear cases.

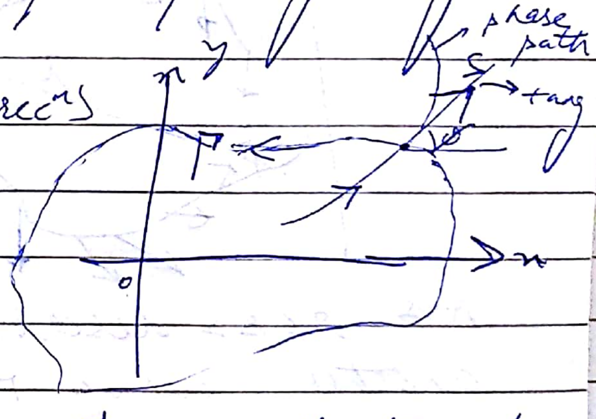
Phase diagram doesn't give info about behaviour of paths at infinity beyond boundaries (Global path).

3.1 The index of a point.

Given the system: $\dot{x} = X(x,y) \quad \dot{y} = Y(x,y) \quad (3.1)$
let Γ be any smooth closed curve consisting of ordinary points of (3.1). Let Q be a point on Γ.
Then there is one and only one phase path passing through Q.

The paths (without implic^n as to direc^n) belong to the family described by the Eqn:

$$\frac{dy}{dx} = \frac{Y(x,y)}{X(x,y)}$$



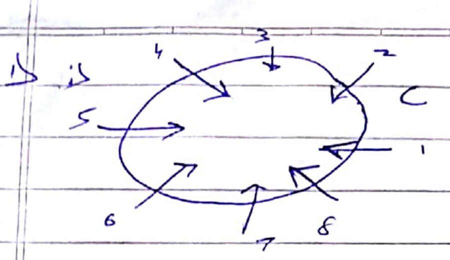
$S = (X, Y)$ is tangential to phase path through the point and points in direc^n of increasing t.

$$\boxed{\tan\phi = Y/X}$$

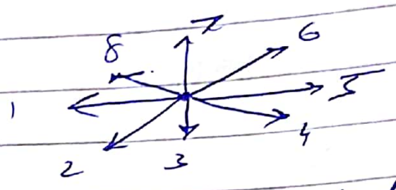
The curve Γ is traversed in the counterclockwise direction and variation of ϕ is followed along it. ϕ will not return to original value but differ by $2\pi n$ after a full cycle.

Example.

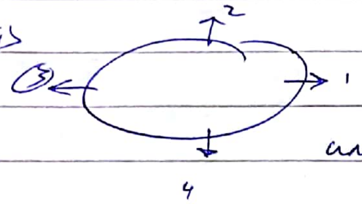
Examples

i) 

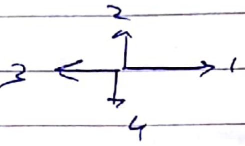
(stable node)



$I_c = +1$ anticlockwise

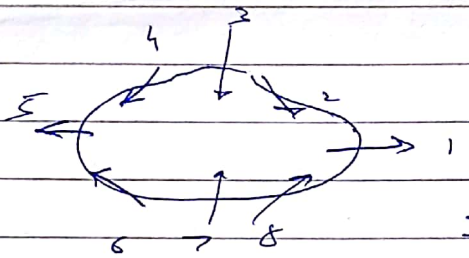
ii) 

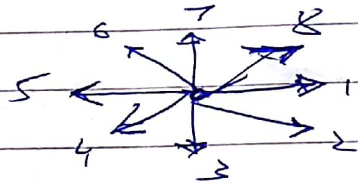
unstable node



$I_c = +1$

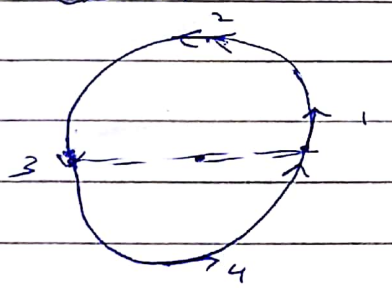
iii) saddle

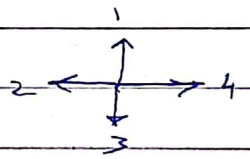




$I_c = -1$

iv) Center

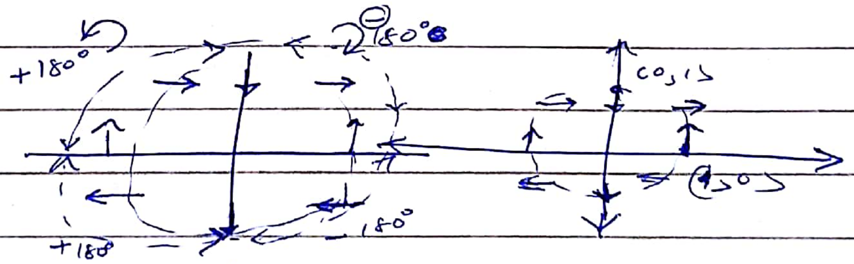




$I_c = +1$

83) $\dot{x} = x^2 y$

$\dot{y} = x^2 - y^2$



$I_c = \cancel{2}$

$x = \cos\theta$ $\quad y = \sin\theta$ $\quad \dot{x} = c^2 s$

$\dot{y} = c^2 - s^2$ $\quad \phi = \tan^{-1} \frac{\cos(2\theta)}{\cos\theta \cdot \sin(2\theta)/2}$

$\tan\phi = \frac{\dot{y}}{\dot{x}} = \frac{2\cos(2\theta)}{2\cos\theta \cdot \cos\theta}$

On unit circle

$\tan\phi \; \tan 2\theta = \frac{2\theta}{\cos\theta}$ $\quad \tan^{-1}\left(\frac{\tan 2\theta \times 2}{\cos\theta}\right)$

$x^2 + y^2 = 1$

(1,0) $\quad$ (0,1) $\quad$ (−1,0) $\quad$ (0,−1)

$\dot{x} = 0$ $\quad \dot{x} = 0$ $\quad \dot{x} = 0$ $\quad \dot{x} = 0$

$\dot{y} = 1$ $\quad \dot{y} = -1$ $\quad \dot{y} = 1$ $\quad \dot{y} = -1$

$\frac{1}{\sqrt{2}}(1, 1) \rightarrow (\dot{x}, \dot{y}) = \frac{1}{2\sqrt{2}}(1, 0)$

$\boxed{I_c = 0}$

Teacher's Sign.:

$\dot{x} = 2x^2 - 1$

$\dot{y} = 2xy$

$\tan\phi = \left(\frac{\sin 2\theta}{\cos 2\theta}\right) = \tan 2\theta$

$I_c = +2\phi$