Summer of Science 2018
Foundations of Quantum Mechanics

Chaitanya Kumar (170260019)

Contents

Abstract

Quantum Mechanics is arguably a very unintuitive theory. My aim in this reading project is to start with the general formulation and then develop a physical understanding of the theory by studying from a perspective of locality and ontology and by studying at an introductory level other interpretations of this theory, namely the Pilot Wave theory, Spontaneous Collapse theory and the Many Worlds interprettion. Prominent physicists of the time were (Einstein and Schrodinger to name a couple) and even some physicists today are bothered by the philosophical implications of quantum mechanics and argue that the theory might be incomplete. A good theory should go beyond just shutting up and doing calculations.

1 Brief Overview of the Basics of Quantum Mechanics

The generalized formulation using the Dirac notation, leading up to wavefunctions, is discussed in this section.

1.1 Mathematical Fundamentals

1.1.1 Ket Spaces and Operators

According to the generalized interpretation of quantum mechanics, the physical state of a quantum particle (or a system of such particles) is represented by a state vector $|\alpha\rangle$ in a Hilbert Space, the dimensionality of which depends on the physical system in consideration. $|\alpha\rangle$ is called a ket and the space to which it belongs is called a 'ket space'. This is in accordance with the Dirac notation. While this vector doesn't have a "real existence" (more on this in subsequent sections), opertions on this vector space with respect to a choice of basis can be used to make predictions about the observables (dynamical quantities) of the system (momentum, energies et cetera).

An observable A is represented by an operator $\hat{A}$. They are generally linear transformations and can take the form of constants or differentials. Solutions to eigenvalue equations yield eigenkets of the observable.

$$\hat{A}|\alpha\rangle = a'|\alpha\rangle \tag{1}$$

These eigenket are complete and form an orthonormal set when the spectrum is discrete.

Any arbitrary ket $|\alpha\rangle$ can be uniquely written as

$$|\alpha\rangle = \sum_{n=1}^{N} c_n |a_n\rangle \tag{2}$$

where c_n are complex coefficients.

1.1.2 Dual Correspondence and Inner Products

Since our physical states exist in a Hilbert Space, there exists a unique 'bra space' dual to every ket space. The general one-to-one dual correspondence goes as

$$c_1|\alpha\rangle + c_2|\beta\rangle \leftrightarrow c_1^*\langle\alpha| + c_2^*\langle\beta| \tag{3}$$

The inner product is then written as

$$\langle\beta|\alpha\rangle \in \mathbb{C} \tag{4}$$

(bra(c)ket)

with the property

$$\langle\alpha|\beta\rangle = \langle\beta|\alpha\rangle^* \tag{5}$$

It follows that

$$\langle\alpha|\alpha\rangle \in \mathbb{R} \tag{6}$$

and

$$\langle\alpha|\alpha\rangle \geq 0 \tag{7}$$

Unless specified otherwise, kets will be assumed to be normalized, i.e.

$$\langle\alpha|\alpha\rangle = 1 \tag{8}$$

Two kets, $|\alpha\rangle, |\beta\rangle$ are said to be orthogonal if

$$\langle\alpha|\beta\rangle = 0 \tag{9}$$

1.1.3 Properties of General Operators

Operators act on kets from the left side and on bras from the right side. Other combinations are illegal products.

$$X.(|\alpha\rangle) = X\,|\alpha\rangle$$
$$(\langle\alpha|).X = \langle\alpha|\,X$$

Operator addition is commutative and associative, operators are closed under linear combinations in a given space and are inn general, not dual to each other.

$$X\,|\alpha\rangle = \langle\alpha|\,X^{\dagger} \tag{10}$$

Operators $XandY$ can be multiplied, but they don't commute in general.

The outer product in a of a bra and a ket is in general another operator and is written as

$$(|\alpha\rangle).(\langle\beta|) = |\alpha\rangle\langle\beta| \tag{11}$$

Products of two bras and the product of two kets are illegal when both elements belong to the same space. The associative axiom of multiplication states that the associative property holds for combinations of the above mentioned products, i.e.

$$(\langle\alpha|\beta\rangle).\,|\gamma\rangle = |\alpha\rangle\,.(\langle\beta|\gamma\rangle) \tag{12}$$

and

$$(\langle\alpha|).(\hat{X}\,|\beta\rangle) = (\langle\alpha|\,\hat{X}).(|\beta\rangle) \equiv \langle\alpha|\hat{X}|\beta\rangle\,. \tag{13}$$

Notice that $\langle\alpha|\beta\rangle$ rotated $|\gamma\rangle$ in the direction of $|\alpha\rangle$.

It follows that

$$\langle\alpha|\hat{X}|\beta\rangle = \langle\beta|\hat{X}^{\dagger}|\alpha\rangle^{*} \tag{14}$$

and for a Hermitian operator

$$\langle\alpha|\hat{X}|\beta\rangle = \langle\beta|\hat{X}|\alpha\rangle^{*} \tag{15}$$

It shall be shown later on that quantum mechanics is mostly concerned with Hermitian operators.

1.1.4 Hermitian Operators and their Eigenkets

Let $\hat{A}$ be a Hermitian operator. The eigenvalues of A must be real and the eigenkets form an orthonormal set.

Proof. Consider the eigenvalue equations

$$\hat{A}\,|\alpha\rangle = a'\,|a'\rangle \tag{16}$$

and since A is Hermitian

$$\langle a''|\,\hat{A} = a''^{*}\langle a''| \tag{17}$$

Multiplying both sides of 16 by $\langle a''|$, both sides of 17 by $|a'\rangle$ on the right and subtracting one of the resulting equations from the other

$$(a' - a''^{*})\,\langle a''|a'\rangle = 0 \tag{18}$$

Consider the case $\langle a''| = |a'\rangle$. Then, $\langle a'|a'\rangle = 1$ and

$$a' = a'^* \Rightarrow a' \in \mathbb{R} \tag{19}$$

Coming back to 18, we have in the case when $\langle a''| \neq |a'\rangle$

$$\langle a''|a'\rangle = 0 \tag{20}$$

$\square$

Therefore, we have

$$\langle a'|a''\rangle = \delta_(a'a'') \tag{21}$$

1.1.5 Eigenkets as Base Kets

Any arbitrary ket in this space can be expanded using the eigenkets of a Hermitian operator, since they form a minimal spanning orthonormal set.

$$|\alpha\rangle = \sum_{a'} |a'\rangle \tag{22}$$

Multiplying $\langle a'|$ on the left and using the orthonormality property, we get

$$c_{a'} = \langle a'|\alpha\rangle \implies |\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\,|\alpha\rangle \tag{23}$$

Which also means that we have found the identity operator

$$\sum_{a'} |a'\rangle\langle a'| = 1 \tag{24}$$

where 1 represents the identity operator and not the scalar. Now, consider the square of the norm of $|\alpha\rangle$, $\langle\alpha|\alpha\rangle$. Using the identity operator.

$$\langle\alpha|\alpha\rangle = (\langle\alpha|).(\sum_{a'} |a'\rangle\langle a'|).(|\alpha\rangle) = \sum_{a'} |\langle a'|\alpha\rangle|^2 \tag{25}$$

This implies that if $|\alpha\rangle$ is normalized

$$\sum_{a'} |c_{a'}|^2 = 1 \tag{26}$$

Also consider the operator $|a'\rangle\langle a'|$ operating on an arbitrary ket

$$(|a'\rangle\langle a'|).\,|\alpha\rangle = |a'\rangle\,\langle a'|\alpha\rangle = c_{a'}\,|a'\rangle \tag{27}$$

$|a'\rangle\langle a'|$ is hence the projection operator along $|a'\rangle$ and is represented by $\Lambda_{a'}$. Equations 26 and 27 are very significant in the general probabalstic interpretation of quantum mechanics.

1.2 Measurement Theory at a Glance

The initial hypothesis is that a physical state of a quantum system is represented by a state vector $|\alpha\rangle$, which can in turn be written in terms of eigenkets of an operator $\hat{A}$ as base kets as

$$|\alpha\rangle = \sum_{a'} |a'\rangle \langle a'|\alpha\rangle \tag{28}$$

On performing a measurement of an observable A, the system is forced into one of the eigenstates of A, i.e.

$$|\alpha\rangle \xrightarrow{A\ measurement} |a'\rangle \tag{29}$$

and the value of the observable is measured to be a'. The probability of the initial state collapsing into the eigenstate $|a'\rangle$ is $|\langle a'|\alpha\rangle|^2$. This means that if masurements of A are performed on a pure ensemble of the state $|\alpha\rangle$, then the probabilty distribution of the eigenvalues obtained will correspond to this result. A sort of confirmation of this postulate can be seen in equation 26. The eigenkets of A form an orthnormal basis and hence are mutually exclusive outcomes. The observer cannot determine before taking a measurement which eigenstate the system will collapse into. This probabalistic interpretation is considered a fundamental postulate of quantum mechanics.

The expectation value of the observable is defined as

$$\langle A\rangle = \langle\alpha|A|\alpha\rangle = \sum_{a'}\sum_{a''} \langle\alpha|a''\rangle \langle a''|\,\hat{A}\,|a'\rangle \langle a'|\alpha\rangle \tag{30}$$

which simplifies to the familiar form of the expression for expected value

$$\langle A\rangle = \sum_{a'} a'|\langle a'|\alpha\rangle|^2 \tag{31}$$

1.2.1 Compatibility of Observables

Observables A and B are said to be compatible if their corresponding opperators obey

$$[\hat{A}, \hat{B}] = 0 \tag{32}$$

else, incompatible. The physical implication is that A and B measureents don't interfere with each other when performed on the same system successively. Assuming that the ket space is spanned by the eigenkets of A as well as by eigenkets of B and choosing the eigenkets of A as our base kets, it can be shown that $\langle a'|\hat{B}|\hat{a''}\rangle$ form a diagonal matrix, i.e.

$$\langle a'|\hat{B}|a''\rangle = \delta_{a',a''} \langle a'|\hat{B}|a'\rangle \tag{33}$$

and more importantly

$$\hat{B}\,|a'\rangle = (\,\langle a'|\hat{B}|a'\rangle).\,|a'\rangle \tag{34}$$

i.e $|a'\rangle$ is a simultaneous eigenket of A and B. $|a', b'\rangle$ can be used to characterize this simultaneous eigenket.

This holds even if there is an n-fold degeneracy,

$$\hat{A}\left|a'^{(i)}\right\rangle = a'\left|a'^{(i)}\right\rangle for\ i = 1, 2, \ldots, n \tag{35}$$

where the n $\left|a'^{(i)}\right\rangle$ form an orthonormal basis with the same eigenvalue a'.

In the general case of multiple commuting operators, or even a maximal set of commuting operators, some operators may have degeneracies, but a collective index K' can be used

$$|K'\rangle = |a', b', \ldots\rangle \tag{36}$$

and the orthonormality relation and the completeness relation are given by

$$\langle K'|K''\rangle = \delta_{K',K''} = \delta_{a',a''}\delta_{b',b''}\ldots \tag{37}$$

and

$$\sum_{K'} |K'\rangle\langle K'| = \sum_{a'}\sum_{b'}\ldots|a', b', c', \ldots\rangle\, a', b', c', \ldots = 1 \tag{38}$$

Now, consider measurements of compatible observables A and B. We measure A and get result a' and then we measure B to get result b'. If we measure A again after this, then the result will be a' with certainty, which means that the second meeasurement does not destroy information obtained (or created, whatever) from the first measurement. When the eigenvalues of A are non-degenerate

$$|\alpha\rangle \xrightarrow{A\ measurement} |a', b'\rangle \xrightarrow{B\ measurement} |a', b'\rangle \xrightarrow{A\ measuremrnt} |a', b'\rangle\,. \tag{39}$$

When there is n-fold degeneracy, the system is thrown into a superposition after the first A measurement

$$|\alpha\rangle \xrightarrow{A\ measurement} \sum_{i}^{n}\left|a', b'^{(i)}\right\rangle \tag{40}$$

where the kets $\left|a', b'^(i)\right\rangle$ have the same eigenvalue a' with respect to A. The second measurement will further collapse the state to one of the $\left|a', b'^(i)\right\rangle$. Subsequent A and B measurements will always yield the results a' and b' respectively.

In the case of incompatible observables, indeterminism and uncertainty relations come into play. They don't have complete sets of simultaneous eigenkets, even though a proper subset of their eigenkets may be simultaneous. Now, consider sequential measurements of A, B and C in that order. The first filter selects some particular $|a'\rangle$ and rejects the others. The next filter selects a particular $|c'\rangle$ and rejects the rest and similarly the third filter selects a $|c'\rangle$. The probability of obtaining a partiular c' is then

$$|\langle c'|b'\rangle|^2 |\langle b'|a'\rangle|^2 \tag{41}$$

Summing over all b', we have the probablity of getting c'

$$\sum_{b'} |\langle c'|b'\rangle|^2 |\langle b'|a'\rangle|^2 = \sum_{b'} \langle c'|b'\rangle \langle b'|a'\rangle \langle a'|b'\rangle \langle b'|c'\rangle \tag{42}$$

Comparing this with the situation in which apparatus for B measurement is absent, we have

$$|a'\rangle = \sum_{b'} |b'\rangle\langle b'| \, |a'\rangle \tag{43}$$

and the probability of obtaining a particular c' is then

$$|\langle c'|c'\rangle|^2 = \left| \sum_{b'} \sum_{b'} \langle c'|b'\rangle \langle b'|a'\rangle \langle a'|b''\rangle \langle b''|c'\rangle \right|, \tag{44}$$

which is different from the previous result (42). The results from C measurements depend on whether B measurements were done or not. Even though from equation 43 it appears as though $|a'\rangle$ is "made up of" the eigenkets $|b'\rangle$ because it can be expanded with those as the base kets. This and other similar results along with entanglement and the collapse postulate result in the quirkiest implications of quantum mechanics. They shall be explored further in section 3 and onwards.

1.2.2 The General Uncertainty Principle

Consider the operator A. We can construct another operator

$$\Delta A = \hat{A} - \langle A \rangle \tag{45}$$

where ΔA^2 is the dispersion. Its expectation value is called the dispersion in A (or standard deviation in the A measurements of a pure ensemble of a given system).

$$\left\langle (\Delta A)^2 \right\rangle = \langle \hat{A}^2 - 2\hat{A} \langle A \rangle + \langle A \rangle^2 \rangle = \left\langle A^2 \right\rangle - \langle A \rangle^2 . \tag{46}$$

If we talk about the dispersion in A for one of its eigenstates, we see that

$$\left\langle A^2\right\rangle - \langle A\rangle^2 = \langle a'|\hat{A}^2|a'\rangle - \langle a'|\hat{A}|a'\rangle^2 = (a')^2 - (a')^2 = 0 \tag{47}$$

Using Scharz's inequality:

$$\left\langle(\Delta A)^2\right\rangle\left\langle(\Delta B)^2\right\rangle \geq |\Delta A\Delta B|^2 \tag{48}$$

The right side can be expanded as:

$$\Delta A\Delta B = \frac{1}{2}[\Delta A, \Delta B] + \frac{1}{2}\{\Delta A, \Delta B\} \tag{49}$$

where the commutator is anti-hermitian and the second term is hermitian. We then have

$$\langle\Delta A\Delta B\rangle = \frac{1}{2}[\Delta A, \Delta B] + \frac{1}{2}\{\Delta A, \Delta B\} \tag{50}$$

The second term on the right is purely imaginary. Now,

$$|\langle\Delta A\Delta B\rangle|^2 = \frac{1}{4}|[\Delta A, \Delta B]|^2 + \frac{1}{4}|\{\Delta A, \Delta B\}|^2 \tag{51}$$

Omitting the second term from the right, we get the general uncertainty result

$$\left\langle(\Delta A)^2\right\rangle\left\langle(\Delta B)^2\right\rangle \geq \frac{1}{4}|\langle[A, B]\rangle|^2 \tag{52}$$

2 Locality and Ontology

The meaning of locality and ontology and their relevance in physical theories are discussed in this section.

2.1 Newtonian Mechanics

This theory presents some of the fundamental laws of the universe as visualized by Isaac Newton. He postulated that the physical realm is made up of solid, impenetrable particles and all physical objects are porous substances formed of these particles. The motion of these particles is governed by the forces (gravity, magnetism and electricity) they exert on each other. While Newton did not have a model for short ranged forces, he had developed a fairl well worked out theory of gravitation.

According to this theory, the universe is visualised as having a Euclidean Geometry the force exerted on the i^{th} particle having mass m_i at a position $\vec{\mathbf{r_i}}$ by the j^{th} particle at $\vec{\mathbf{r_j}}$ having mass m_j is given by

$$\vec{\mathbf{F}}_{i,j} = \frac{Gm_i m_j}{r_{i,j}^2}\hat{r}_{i,j} \tag{53}$$

where

$$\vec{\mathbf{r}}_{i,j} = \vec{\mathbf{r}}_j - \vec{\mathbf{r}}_i \tag{54}$$

The total force on the i_{th} particle is then given by

$$\vec{\mathbf{F}}_i = \sum_{j \neq i} \vec{\mathbf{F}}_{i,j} \tag{55}$$

The trajectory of the particle can then be calculated using Newton's second law.

$$\vec{\mathbf{F}}_i = m_i \vec{\mathbf{a}}_i = m_i \ddot{\vec{\mathbf{r}}}_i \tag{56}$$

The important fact to note here is that the force on any particle depends on the *instantaneous positions* of all other particles. The gravitational force does not have a finite propogation speed and is hence *non-local*; it exhibits spooky action at a distance. Newton himself did not believe that his formulation of gravitation was a complete one. He expressed his concerns in a letter to Richard Bentley:

> It is inconceivable that inanimate brute matter should, without the mediation of something else which is not material, operate upon and affect other matter without mutual contact...That gravity should be innate, inherent, and essential to matter, so that one body may act upon another at a distance through a vacuum, without the mediation of anything else, by and through which their action and force may be conveyed from one to another, is to me so great an absurdity that I believe no man who has in philosophical matters a competent faculty of thinking can ever fall into it.

Newton himself did not regard this formulation as a complete theory, but merely a starting point, and his philosophical instincts were proven correct by the subsequent development of the classical field theories and finally the special and general theories of relativity. Newtonian mechanics is hence, losely speaking, a stark violation of relativity. In more concrete terms, relative local causality is not obeyed in Newtonian mechanics and because of whivh there exists a dynamically priviledged frame of reference which can be used to compute the trajectories and equations of motion of the system, which contradicts the fundamental principle of relativity.

2.2 Maxwellian Electrodynamics

Here is a theory of electrically charged particles interacting with electric and magnetic fields - an example of a local classical theory. According to *Coulomb's Law*, which is analogous to Newtonian graviation:

$$\vec{\mathbf{F}}_{i,j} = -\frac{q_i q_j}{4\pi\epsilon_0 r_{i,j}^2}\hat{r}_{i,j} \tag{57}$$

and the net force is the sum over all the force due to all charge elements in the universe and Newton's second law can be used to compute the trajectories of charged particles. This naïve interpretation of Electrostatics leads one to believe that this theory is exactly the same as Newtonian gravitation, but with different physical quantities. This changes radically when classical fields are introduced into the theory. We can, instead of *Coulomb's law*, write

$$\vec{\mathbf{E}}(\vec{\mathbf{r}}) = \sum_{i=1}^{n} \frac{kq_i}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}_i|^2}\hat{r}_i \tag{58}$$

This equation encapsulates both Coulomb's law and the superposition principle. The net force on a charge of magnitude q located at $\vec{\mathbf{r}}$ is then given by

$$\vec{\mathbf{F}} = q\vec{\mathbf{E}}(\vec{\mathbf{r}}) \tag{59}$$

and the general expression for force on a charge element is

$$\vec{\mathbf{F}} = q(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}}) \tag{60}$$

The electric field and its related magnetic field are not merely mathemaical constructs that aid in calculation, they are actual physical entities in this theory; just like matter, they are capable of carrying energy and momentum and have other properties that make them "physically real".

Now, consider Maxwell's equations

$$\vec{\nabla}.\vec{\mathbf{E}} = \frac{\rho}{\epsilon_0} \tag{61}$$

$$\nabla.\vec{\mathbf{B}} = 0 \tag{62}$$

$$\boldsymbol{\nabla} \times \vec{\mathbf{B}} = \mu_0\vec{\mathbf{j}} + \mu_0\epsilon_0\frac{\partial}{\partial t}\vec{\mathbf{E}} \tag{63}$$

$$\boldsymbol{\nabla} \times \vec{\mathbf{E}} = -\frac{\partial}{\partial t}\vec{\mathbf{B}} \tag{64}$$

Taking the curl of 64 and using the identity $\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \vec{\mathbf{V}} = \nabla(\nabla.\vec{\mathbf{V}}) - \nabla^2\vec{\mathbf{V}}$, we can solve to get

$$\nabla^2\vec{\mathbf{E}} - \frac{1}{c^2}\frac{\partial^2\vec{\mathbf{E}}}{\partial t^2} = \nabla(\frac{\rho}{\epsilon_0}) + \frac{\partial}{\partial t}(\mu_0\vec{\mathbf{j}}). \tag{65}$$

Similarly,

$$\nabla^2\vec{\mathbf{B}} - \frac{1}{c^2}\frac{\partial^2\vec{\mathbf{B}}}{\partial t^2} = -\mu_0\boldsymbol{\nabla} \times \vec{\mathbf{j}}, \tag{66}$$

where $c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$. Both of these partial differential equations are of the familiar wave equation form. In empty space, $\rho = 0$ and $\vec{\mathbf{j}} = 0$, and we get the equations for electromagnetic waves freely popogating with speed c. Now, consider the general wave equation:

$$\nabla^2\Psi(\vec{\mathbf{r}}, t) - \frac{1}{c^2}\frac{\partial^2\Psi(\vec{\mathbf{r}}, t)}{\partial t^2} = f(\vec{\mathbf{r}}, t) \tag{67}$$

where f is the source term and is hence some time or space derivative of charge or current density. Let the source e localised at a point in spacetime, i.e. $f(\vec{\mathbf{r}}, t) = \delta^3(\vec{\mathbf{r}} - var')\delta(t - t')$ (General sources can be considered as the sum/integral over such source elements and the net effect can be evaluated using the superposition principle). Then

$$\Psi_{\vec{\mathbf{r}}',t'}(\vec{\mathbf{r}}, t) = -\frac{1}{4\pi}\frac{\delta\left(t - \left[t' + \frac{\left|\vec{\mathbf{r}}-\vec{\mathbf{r}}'\right|}{c}\right]\right)}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}}'\right|}, \tag{68}$$

is the particular solution. The general solution will include the complemntary part, which is just the free electromagnetic wave funtion. which means that the field has non-zero magnitude only at points in spcetime where the argument in the delta-funcion is zero., i.e. only at positions $\vec{\mathbf{r}}$ and times t that can be reached by the field from the source at $\vec{\mathbf{r}}'$, t' at speed c. Thus, causal influences in this theory, namely the electric and magnetic fields, after originating from a source can propogate no faster than the speed of light in vacuum.

Clearly, it is established that Maxwellian Electrodynamics is a local theory and the electric and magnetic fields are local variables. This situation can be represented in an $x - y - t$ spacetime diagram (Figure 1). The point source in considertion appears at $(\vec{\mathbf{r}}', t')$ and its causal influence is depicted by the future light cone. Only the entities lying on or inside this light cone may interact with the causal influence (entities inside will indirectly interact

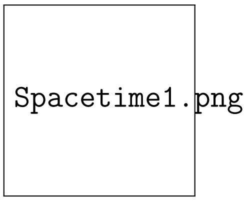

Figure 1: The source $f(\vec{\mathbf{r}}, t) = \delta^3(\vec{\mathbf{r}} - \vec{\mathbf{r}}')\delta(t - t')$

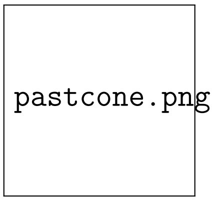

Figure 2: The past light cone of an event

with the causal influence). In another such spacetime diagram consider a charged particle at $(\vec{\mathbf{r}}, t)$. Again, because of relative local causality, it will be influenced only by events lying on or inside its past light cone; the motion of the particle at $(\vec{\mathbf{r}}, t)$ depends on $\vec{\mathbf{E}}(\vec{\mathbf{r}}, t)$ and $\vec{\mathbf{B}}(\vec{\mathbf{r}}, t)$, which in turn are influeced by source terms and (from the complementary part) and background electromagnetic radiation on the past light cone.

2.3 Bell's Formulation of Locality

Consider an event $\chi(\vec{\mathbf{r}}, t)$ at $(\vec{\mathbf{r}}, t)$, its past light cone and a hypersurface at time $t' < t$. We label the portion of the plane on and inside the light cone, which will be a circle of radius $c(t - t')$, as Σ. A complete description of the of the state of all fields and charges on Σ will determine what happens at χ (or in the case of non-deterinistic theories like quantum mechanics, determine the possibilities and their probability distribution at χ). Any additional information must be redundant in a strictly local theory. Formally, this can be written as:

$$\chi(\vec{\mathbf{r}}, t) = f(C_\Sigma) \tag{69}$$

where C_Σ is a complete description of events at Σ. In general, for we need to consider non-deterministic theories, deterministic ones being the unit probabality special cases:

$$P[\chi_1 | C_\Sigma] = P[\chi_1 | C_\Sigma, \chi_2], \tag{70}$$

i.e. the probability for some physical even χ_1 of happenning at point 1, given a complete description of events at on Σ is the same as its probability if in

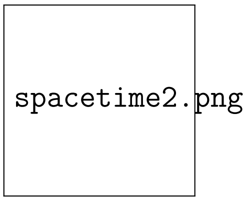

Figure 3: The spacetime diagram for Bell's formulation of locality

addition an event χ_2, that cannot be causally influenced by the events in Σ.

2.4 Ontology

"Ontology" can be primitively understood as the philosophical study of existence - what it means for something to exist. For example in Newtonian Mechanics, it is fairly trivial that constituent particles that are influenced by forces exist physically. Similarly, Maxwellian electrodynamics supplements those particles with the classical fields.

Now, consider the potential formulation of Electrodynamics as a distinction between ontological and epistomological entities:

$$\vec{\mathbf{B}} = \boldsymbol{\nabla} \times A \tag{71}$$

Faraday's law can thus be rewritten as

$$\boldsymbol{\nabla} \times \left(\vec{\mathbf{E}} + \frac{\partial \vec{\mathbf{A}}}{\partial t} \right) = 0 \tag{72}$$

This will be true if we introduce a scalar function ϕ as:

$$\vec{\mathbf{E}} + \frac{\partial \vec{\mathbf{A}}}{\partial t} = -\nabla \phi. \tag{73}$$

The electric field can then be written as

$$\vec{\mathbf{E}} = -\nabla \phi - \frac{\partial \vec{\mathbf{A}}}{\partial t} \tag{74}$$

Rewriting Maxwell's equations

$$\nabla^2 \phi + \frac{\partial}{\partial t}(\vec{\nabla}.\vec{\mathbf{A}}) = -\frac{\rho}{\epsilon_0} \tag{75}$$

and

$$\nabla^2 \vec{\mathbf{A}} - \frac{1}{c^2}\frac{\partial^2 \vec{\mathbf{A}}}{\partial t^2} = -\mu_0 \vec{\mathbf{j}} + \vec{\nabla}(\vec{\nabla}.\vec{\mathbf{A}} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}) \tag{76}$$

We can also exploit gauge freedom, that is $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ will remain unchanged if we make the following gauge transformations:

$$\vec{\mathbf{A}} \rightarrow \vec{\mathbf{A}} + \vec{\nabla}\lambda \tag{77}$$

$$\phi \rightarrow \phi - \frac{\partial \lambda}{\partial t} \tag{78}$$

We can choose a particular set of potentials according to convenience. In the Lorrentz gauge, the condition satisfied is:

$$\vec{\nabla}.\vec{\mathbf{A}} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} \tag{79}$$

and the wave eqations satisfied by the potentials then become

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon_0} \tag{80}$$

and

$$\nabla^2 \vec{\mathbf{A}} - \frac{1}{c^2}\frac{\partial^2 \vec{\mathbf{A}}}{\partial t^2} = -\mu_0 \vec{\mathbf{j}} \tag{81}$$

Clearly, the effects of charges on potential travel outward at light speed. But, consider the Coulomb gauge:

$$\vec{\nabla}\vec{\mathbf{A}} = 0 \tag{82}$$

This implies the following potential wave equations:

$$\nabla^2 \vec{\mathbf{A}} - \frac{1}{c^2}\frac{\partial^2 \vec{\mathbf{A}}}{\partial t^2} = -\mu_0 \vec{\mathbf{j}} + \frac{1}{c^2}\vec{\nabla}\frac{\partial \phi}{\partial t} \tag{83}$$

and

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0} \tag{84}$$

The last equation is the same as equation 80, but with the propogation speed set to infinity. ϕ is hence a non-local entity in the Coulomb gauge, i.e. it changes instantaneously with changes in configuration of distant charges. But, one shouldn't be bothered by this apparent non-locality, because ϕ and $\vec{\mathbf{A}}$ are only mathematical constructs meant to aid in calculation and not "physically real". A more formal way of putting it - the event χ can be, for example, the the value of $\vec{\mathbf{E}}$ or $\vec{\mathbf{B}}$, but not ϕ or $\vec{\mathbf{A}}$.

Consideration of the ontological status of a theory should be established and is generally not trivial. At this point, we might say that for something to exist ontologically, it must have a description in the framework of spacetime. However, ontology in quantum mechanics not trivial at all, as shall be seen in subsequent sections.

3 The Measurement Problem

Measuremeent plays a central role in quantum mechanics. While measurements in classical systems are fairly starightforward to describe formally. A formal description in quantum mechanics requires a lot of additional considerations. For now, the term"measuring device" implies a probe connected to a black box that has a movable pointer in front of a calibrated background and conceals the mechanism that provides a causal link between the physical quanitity being measured and its effect on the pointer. This might seem too specialized, but a lot of measuring devices work using the same principle and this setup captures the essential features of an acceptable measuring device.

However, this suggests that there is an apparent dichotomy in the nature of the universe according to the quantum picture; there is the quantum realm wherein eveerything is governed by quantum dynamics, but translation and interpretation of a measurement of systems in this realm necessitates the use of a classical picture. Ideally, as a complete theory, quantum mechanics should be able to provide a complete picture of the universe, including the outcome and final interpretation of measurements, without resorting to a decidedly non-quantum picture that has completely different ontologies and dynamical laws.

While the wave function of a system evolves according to Schrödinger's equation when not under observation, it fails momentrily to obey this evolution during a *measurement* process; It becomes imperative to define what a measurement process is and how it differs from other quantum processes that don't cause a discontinuous collapse in a superposition state.

An attempt at a formal description of pointers can be made as follows:

A particle in a one dimensional box starts off with the state

$$\psi_0(x) = c_1\psi_1(x) + c_2\psi_2(x) + c_3\psi_3(x) \tag{85}$$

Let the pointer be a free particle schematically represented by the gaussian wavepacket centered at y_0, its wavefunction given by

$$\phi(y) = Ne^{\frac{(y-y_0)^2}{4\sigma^2}} \tag{86}$$

At the end of an energy measurement, the pointer should settle at a position that is at a distance proportional to the result of the energy measurement, E_n. Let the measurement begin at $t = 0$. The joint wavefunction of the pointer and the particle will be at this moment

$$\Psi_0(x, y) = \psi_0(x)\phi(y) \tag{87}$$

The state then evolves according to Schrödinger's equation.

$$\iota\hbar\frac{\partial}{\partial t}\Psi(x,y,t) = \hat{H}\Psi(x,y,t) \tag{88}$$

The Hamiltonian over here has three parts. The hamiltonian for the particle confined in the box whose degree of freedom is x.

$$\hat{H}_x = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) \tag{89}$$

The second part is the hamiltonian corresponding to the kinetic energy of the pointer

$$\hat{H}_y = -\frac{\hbar^2}{2M}\frac{\partial^2}{\partial y^2} \tag{90}$$

The last term in the hamiltonian corresponds to the interaction energy of the interaction between the pointer and the particle:

$$\hat{H}_{int} = \lambda\hat{H}_x\hat{p}_y = -\iota\hbar\lambda\hat{H}_x\frac{\partial}{\partial y} \tag{91}$$

where λ is a constant. Now, our pointer must be a heavy one in order for it to "settle" with precision. M is hence large and the energy contribution of the pointer's hamiltonian is small, allowing reasonably the approxiamtion $\hat{H}_y = 0$.

Putting this in action, suppose we have initially the particle starting out in an energy eigenstate ψ_n

$$\Psi(x,y,0) = \psi_n(x)\psi(y) \tag{92}$$

Evolving according to Scrödinger's equation

$$\iota\hbar\frac{\partial\Psi}{\partial t} = (\hat{H}_x + \hat{H}_y)\Psi \tag{93}$$

For the sake of simplicity, we let the interaction energy be very large, so that we can neglect any other contribution. Besides, $\hat{H}_x$ would only add a phase factor to the solution. We now have

$$\frac{\partial\Psi}{\partial t} = -\lambda E_n\frac{\partial\Psi}{\partial y} \tag{94}$$

The solution is then

$$\Psi(x,y,t) = \psi_n(x)\psi(y - \lambda E_n t) \tag{95}$$

At the end of the interaction, say at $t = T$, we have state of the prticle pointer system is evidently

$$\Psi(x, y, t) = \psi_n(x)\psi(y - \lambda E_n T) \tag{96}$$

i.e. the particle is in the n_{th} energy eigenstate and the pointer is centred at a distance proportioal to the energy E_n of this eigenstate. This schematic model works, at least on paper, as intended. The pointer is indeed (apparenty) useful for measuring the energy of the particle. But, what if our particle starts out in the most general superposition state? We have

$$\Psi(x, y, 0) = \left(\sum_i c_i\psi_i\right)\phi(y) \tag{97}$$

Alas, the hamiltonian, and quantum mechanical operators in general, are linear and without the collapse postulate, we get this absurd post measuremet state:

$$\Psi(x, y, T) = \sum_i c_i\psi_i(x)\phi(y - \lambda E_i T) \tag{98}$$

which is an entangled superposition. The pointer is infected with the indeterminism.

4 The Locality Problem and the EPR Paradox

Podolsky wrote the most famous criticisms (after discussions with Einstein and Rosen) of the Copenhagen interpretation of the Quantum theory. Their main argument was that if the quantum theory is complete, locality must be violated and if locality is held sacred, then the theory can't be complete and there must be completed by introducing local hidden variables. The abstract of the paper reads:

> n a complete theory there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system. In quantum mechanics in the case of two physical quantities described by non-commuting operators, the knowledge of one precludes the knowledge of the other. Then either (1) the description of reality given by the wave function in quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality. Consideration of the problem

> of making predictions concerning a system on the basis of measurements made on another system that had previously interacted with it leads to the result that if (1) is false then (2) is also false. One is thus led to conclude that the description of reality as given by a wave function is not complete. [?]

The overall goal of this paper was to establish the existence of more physically real entities than there are counterparts in quantum descriptions. EPR claim that it should be possible to bypass the uncertainty relations and hence establish the simultneous existence of two observables with non-commuting operators (say position and momentum). They had in particular a beef with the interpretation of measurements of entangled systems.

Consider a system of two entangled particles that have been separated spatially. Let their degrees of freedom post separation be x_1 and x_2. Let the entangled state be

$$\Psi(x_1, x_2) = \delta(x_1 - x_2) = \int \delta(x - x_1)\delta(x - x_2)dx \tag{99}$$

In the two dimensional x_1-x_2 configuration space this appears as a sharp ridge along the line $x_1 = x_2$, which is basically a superposition of position eigenstates in which both particles are located at x, i.e. their postions are entangled and there is hence a definite correlation. Information about the position of any one of the particles gives a direct implication about the other one. The implication here is that collapsing one of the particles by measuring its position must lead to an indirect collapse of the second particle and this must happen *instantaneously*, which means that quantum theory is decidedly $non - local$. This can also be shown using simpler entangled spin states. Alternatively, if validation of locality is assumed, then the particle already had a postition priot to measuement, but quantum theory can't predict it and must hence be incomplete.

More formally, EPR further rewrite the entangled state in another form

$$\Psi(x_1, x_2) = \delta(x_1 - x_2) = \frac{1}{2\pi}\int e^{\iota k(x_1 - x_2)}dk = \frac{1}{2\pi}\int e^{\iota k x_2}e^{-\iota k x_2}dk \tag{100}$$

which is a superposition over all possible values of k, where the momentum eigenvalues are given by $p = \hbar k$. Moreover, the momenta p_1 and p_2 corresponding to positions x_1 and x_2 are perfectly anti-correlated, i.e. $p_1 = -p_2$. This is bascally the same consequence as with position measurements. It follows from this that *assuming relatvistic local causality must be a feature for a physical theory to be valid, both particles must have definite momenta* *and positions simultaneously before measurment*, and quantum theory simply fails to provide a complete picture. There is yet another problem with locality when we look at it formally. If we try to apply Bell's formultion of locality to a system of two spatially separated spin-entangled particles; Events A and B are measurements of their spin in the same direction and C is a hypersurface at some past time that isolates from A the light cone of B:

$$P[A|C_\Sigma] = P[A|C_\Sigma, B] \tag{101}$$

But, what then is C_Σ? One might be tempted to say that it is the state/wavefunction of particle A at that time. But, in an entangled state, what is the state of A? It can't be pealed off from the entire state.

A modification in the definition of locality to overcome this problem would be to take all of C instead of just C_Σ, i.e. we test whether the equality. C in this case has to be the wavefuction of the entangled system as it is claime that it has all the information about the system.

$$P[A|\Psi] = P[A|\Psi, B] \tag{102}$$

holds true or not. The result of A can be up or down with 0.5 probability of each. If the result of B is given, then the outcome of A is known with certainty and B hence provides additional information about the outcome at A, which contradicts the asumption that Ψ provides complete information about the system.

Thus, quantum mechanics and relativistic local causality are inherently incompatible.

5 The Ontology Problem

Quantum mechanics does not seem clear in its ontology. Physically real entities and existing quantities come into play only during measurements. What happens between the source and the measurement apparatus is described using a complex wave function in configuration space and not even necessarily real space. Before Born's statistical interpretation, which treats the Ψ function as a mental abstraction, was established, Schrödinger attempted unsuccessfully to interpret $|\Psi|^2$ as the charge density or mass density distribution. The ontological status of the Ψ function as a probabilty distribution function that describes the behaviour of physical particles is hence vague at best. Heisenberg and those who share his idea of physical reality go as far as saying that what a particle does between measurements is not merely unknowable, but non-existent altogether.

That *does not necessarily imply* that the particle is *not real* between observtions. Some philosophies of reality claim that reality is confined to perception. Kantian philosophy, on the other hand, claims that our perception of reality is limited by the physiological processes involved in perceiving and interpreting things. A particle in a state of superposition outside our perception. On the other hand, observation of said paritcle is indirect perception; the results of measurement and with them the particle itself "spring into physical existence". Ψ essectially describes something that *eventually becomes* a physical entity. As Heisenberg put it, "a strange kind of physical reality just in the middle of possibility and reality."

Some even argue that since quantum mechanics is not clear about its ontology, saying that it is not a local local theory does not necessarily imply that it is cleanly diagnosable as non-local either, because locality means that the causal influences that objects, moving and interacting in spatial three dimensionaal real space, exert on one another always propogate at the speed of light or slower. A theory that does not provide a ceherent local ontology in the first place doesn't rise to this question.

6 The Copenhagen Interpretation

The Copenhagen interpretation in about how the quantum theory discussed thus far should be interpreted in order to work around the absolutely whack ontology. Niels Bohr and Werner Heisenberg were the primary proponents of this theory and Bohr took it upon himself to rebutt the EPR argument. While he was very successful in defending his philosophy, it wasn't until 1964 that local hidden variable theories as suggested by EPR are impossible.

A summary of Bohr's philosophy:

> he quantum theory is characterized by the acknwledgement of a fundamental limitation in the classical physical ideas when applied to atomic phenomena. The situation thus created is of a peculiar nature, since our interpretation of the experimental material essentially rests upon the classical concepts. Notwithstandng the difficulties which hence are involved with the formulation of the quantum theory, it seems as we shall see, that its essence may be expressed in the so-called 'quantum' postulate, which attributes to any atomic process an essential discontinuity, or rather individuality, completely foreign to classical theories and symbolized by Planck's quantum of action. [**?**]
>
> This quantum postulate implies a renunciation as regards the

> causal space-time coordination of atomic processes. Indeed, our usual description of physical phenomena is based entirely on the idea that the phenomena concerned may be observed without disturbing them appreciably. This appears, for example, clearly in the theory of relativity, which has been so fruitful for the elucidation of the classical theories. As emphasised by Einstein, every observation or measurement ultimately rests on the coincidence of two independent; events at the same space-time point. Just these coincidences will not be affected by any differences which the space-time co-ordination of different observers otherwise may exhibit. Now the quantum postulate implies that any observation of atomic phenomena will involve an interaction with the agency of observation not to be neglected. Accordingly, an independedent reality in the ordinary physical sense can neither be ascribed to the phenomena nor to the agencies of observation. After all, the concept of observation is in so far arbitrary as it depends upon which objects are included in the system to be observed. Ultimately every observation can of course be reduced to our sense perceptions. The circumstance, however, that in interpreting observations use has always to be made of theoretical notions, entails that for every particular case it is a question of convenience at what point the concept of observation involving the quantum postulate with its inherent irrationality' is brought in.

So, according to Bohr, quantum mechanics simply does not allow for description of physical phenomena in the framework of space and time, which is why dealing in terms of abstract algebraic is the only choice. The very act of measurement disturbs the system in some way and it is hence not reasonable to ask what was before such a measurement. He draws a very nice parallel between relativity and quantum mechanics in the role of the observer in both theories. While observation in relativity and quantum mechanics don't refer to the same exact thing, Bohr noted that the observation of an event in relativity requires light from an event to reach the observer ("every observation or measurement ultimately rests on the coincidence of two independent; events at the same space-time point"), which is why there can't be an independent reality to the event, i.e. there is no dynamically privileged frame of reference. Observations in quantum mechanics rely on decoherence or interaction of at least two systems, namely the system of particles and the measurement apparatus, with each other. The independent existence of either system at the quantum level is, as some like to say, out of our perception, or simply non-existent.

Scrödinger's wave mechanics really gets us no closer to the spacetime description of things. Wave mechanics was an attempt to remove some apparently irrational ideas introduced by quantum mechanics by trying to make it analogous to the classical description ($x(t)$ verses $\Psi(x,t)$). But, even the wavefunction is just a complex funcition with a configuration-space domain, and not even a real space domain. It is every bit just as abstract as matrix mechanics.

Heisenberg addressed the alleged arbitriness in the Schizophrenic division of the world into observer and system:

> t has been said that we always start with a division of the world into an object, which we are going to study, and the rest of the world, and that this division is to some extent arbitrary. It should indeed not make any difference in the final result if we, e.g., add some part of the measuring device or the whole device to the object and apply the laws of quantum theory to this more complicated object. It can be shown that such an alteration of the theoretical treatment would not alter the predictions concerning a given experiment. This follows mathematically from the fact that the laws of quantum theory are for the phenomena in which Planck's constant can be considered as a very small quantity, approximately identical with the classical laws. But it would be a mistake to believe that this application of the quantum theoretical laws to the measuring device could help to avoid the fundamental paradox of quantum theory. The measuring device deserves this name only if it is in close contact with the rest of the world, if there is an interaction between the device and the observer. Therefore, the uncertainty with respect to the microscopic behavior of the world will enter into the quantum-theoretical system here just as well as in the first interpretation. If the measuring device would be isolated in the terms of classical physics at all. from the rest of the world, it would be neither a measuring device nor could it be described. Certainly quantum theory does not contain genuine subjective features, it does not introduce the mind of the physicist as a part of the atomic event. But it starts from the division of the world into 'object' and the rest of the world, and from the fact that at least for the rest of the we use the classical concepts in our description. This division is arbitrary and historically a direct consequence of our scientific method. [**?**]

Essentially this means that a departure from the quantum theory and the

Copenhagen interpretation would require one to completely abandon concepts that are a consequence of classical mechanics, even for interpretaation of measurements. This, Heisenberg et al argue, should not be done because we are, after all, only human. Our minds are hard wired to directly interpret geometry and not algebraic abstractions.

7 References

- Foundations of Quantum Mechanics: An Exploration of the Physical Meaning of Quantum Theory: Travis Norsen
- Introduction to Quantum Mechanics: David J. Griffiths
- Modern Quantum Mechanics: J. J. Sakurai