SUMMER OF SCIENCE-2019

APPLICATION OF DIFFERENTIAL GEOMETRY IN STRUCTURAL GEOLOGY

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OBJECTIVE OF THIS STUDY-

There is obvious need to describe the complex shapes of curved lineations and surfaces, and the attractiveness of accomplishing this in a quantitative manner. Therefore it is important to study the principles and methods of differential geometry that appear to have the greatest potential for application to structural geology. Differential geometry provides the mathematical constructs to characterize three-dimensional structures via well-defined geometric parameters that can be used in continuum mechanical models of tectonic processes, to describe the departure of geological lineations from a straight line and geological surfaces from a plane.

TOPICS COVERED:

PHASE 1

It consisted of learning basic terms and mathematical tools required for further understanding. It includes the following:

1. Vectors – it is used to describe physical quantity which has both a magnitude and direction associated with it. Various operations associated with vectors are cross product, dot product, scalar triple product, vector triple product.
2. Function – assignment which assigns to each element of a set A a unique element of a set B. f: A→B.
3. Inner product - An inner product is a generalization of the dot product. In a vector space, it is a way to multiply vectors together, with the result of this multiplication being a scalar. More precisely, for a real vector space, an inner product $\langle \cdot, \cdot \rangle$ satisfies the following four properties. Let u, v, and w be vectors and α be a scalar, then:

   1. $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$.

   2. $\langle \alpha\, v, w\rangle = \alpha \langle v, w\rangle$.

   3. $\langle v, w\rangle = \langle w, v\rangle$.

   4. $\langle v, v\rangle \geq 0$ and equal if and only if $v = 0$.

Inner product space: A vector space together with an inner product on it is called an ***inner product space.***

Examples of inner product spaces include:

1. The real numbers $\mathbb{R}$, where the inner product is given by

$$\langle x, y\rangle = x\, y. \tag{1}$$

2. The Euclidean space $\mathbb{R}^n$, where the inner product is given by the dot product

$$\langle (x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)\rangle = x_1\, y_1 + x_2\, y_2 + \cdots x_n\, y_n \tag{2}$$

3. The vector space of real functions whose domain is an closed interval $[a, b]$ with inner product

$$\langle f, g\rangle = \int_a^b f\, g\, d\, x.$$

Metric space – built on the concept of "distance" between objects it is the generalization of the familiar distance between two points on the real line. METRIC is a real valued function.

d(.,.) : X*X → R
A set X with an associated metric is called ***Metric space***.

Tensor- a mathematical object analogous to but more general than a vector, represented by an array of components that are functions of the coordinates of a space. Examples to understand:

1. Suppose that F is an operator which transforms every vector into its mirror image with respect to a given plane. F transforms a vector into another vector and the transformation is linear. Hence F is a second order tensor.
2. The combination (ux) linearly transforms a vector into another vector, hence it is second order tensor.

A thumb rule being followed here is-

First order tesor maps first order tensor to give zeroth order tensor(scalar).
Second order tensor maps first order tensor to give first order tensor.
Third order tensor maps first order tensor to give second order tensor.

Dyad – tensor product itself is a tensor of order 2 called ***dyad***.

a.b- scalar(zeroth order tensor)
axb- vector(first order tensor)

$\mathbf{a} \otimes \mathbf{b}$ - dyad (second order tensor)

Manifold – a collection of points forming a certain kind of set, such as those of a topologically closed surfaces or an analogue of this in 3 or more dimension.

Submanifold – a subset M of R^N is a 'k'- dimensional submanifold if for every point x in M there exists a neighbourhood V of x in R^N, an open set U subset R^k and a smooth map f: U → R^N such that f is continuous and one-one , onto M V and inverse also continuous.

PHASE 2

2.1 CONCEPT AND DESCRIPTION OF LINEATIONS

A primary task for structural geologists is to describe and characterize such surfaces and this maybe accomplished in a mathematically rigorous and complete manner using concepts and tools from differential geometry, the branch of mathematics that brings power to vector calculus to geometry. This phase includes understanding lineations using rigorous differential calculus; also some basic geology terms as follows:

1. Foliation – refers to the repetitive layering in metamorphic rocks due to shearing forces or differential pressure. Each layer can be as thin as sheet of paper or a meter in thickness.
2. Lineation – any linear feature or element in rock. Lineations are the one dimensional counterpart of foliations.
3. Stereographic projections – particular mapping defined on entire space (except projection point) that projects a sphere onto a plane. Wherever defined it is bijective and smooth and preserves angles at which curves meet. It is neither isometric nor area-preserving.
4. Plunge – vertical angle between horizontal and axis(line of maximum elongation of a feature). Plunge measured along the axis of a fold whereas dip is measured along the limbs. It is used to describe tilt of lines and dip for planes.
5. Dip – slope of a geological surface. Direction of dip is compass direction towards which the plane slopes.
6. Slickenline – term employed for fracture surfaces in rocks that are polished and/or coated with secondary mineral growths and that commonly bear a linear structure. They are shear fractures.

TYPE OF LINEATIONS

Some lineations are defined by the intersection of two geological surfaces that separate one volume of rock from another. For example, a fault (F-F') separates the relatively young and undeformed rocks of a sedimentary basin from the older and more deformed sedimentary or metamorphic rocks of the adjacent mountain range. An igneous contact (I-I’) separates older deformed sedimentary or metamorphic rocks from the younger rocks of an igneous intrusion. An angular unconformity (U-U’) separates older sedimentary or metamorphic rocks from the overlying sedimentary strata.

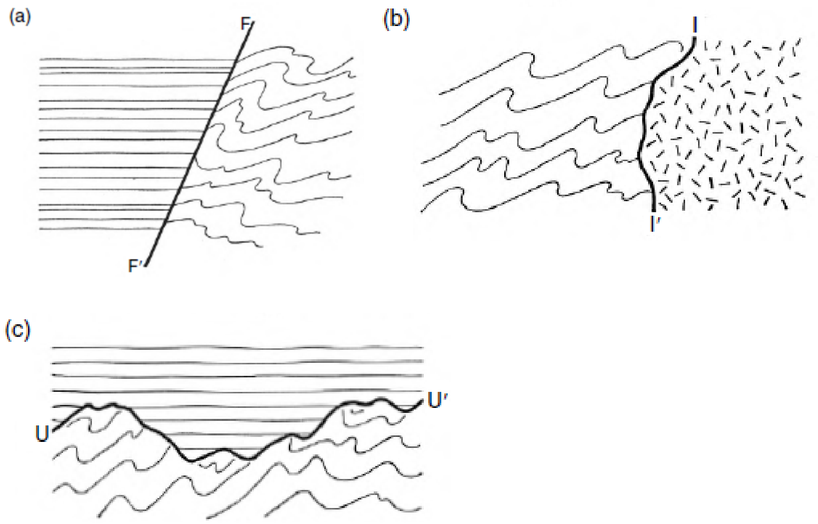

Fig 2.1: Two distinct rock volumes separated across geological surfaces including: (a) a fault, F–F’; (b) an igneous contact, I–I’; and (c) an angular unconformity, U–U’. Intersections of sedimentary or metamorphic layers with these surfaces define discrete lineations.

Consider two roughly planar and continuous geological surfaces, say an igneous contact and an unconformity that intersect one another. To the extent that the surfaces are planar, the intersection defines a straight line. At points along the intersection the attitudes, measured as the plunge direction and plunge, would be approximately equal and the resulting points would plot in a very tight cluster on a stereogram. The type of lineation we have just described is called a ***discrete lineation*** because it is made up of the set of points common to two discrete surfaces. Discrete lineations defined by intersections of geological surfaces- straight lineation at intersection of two planar surfaces and curved lineation at intersection of curved surfaces.

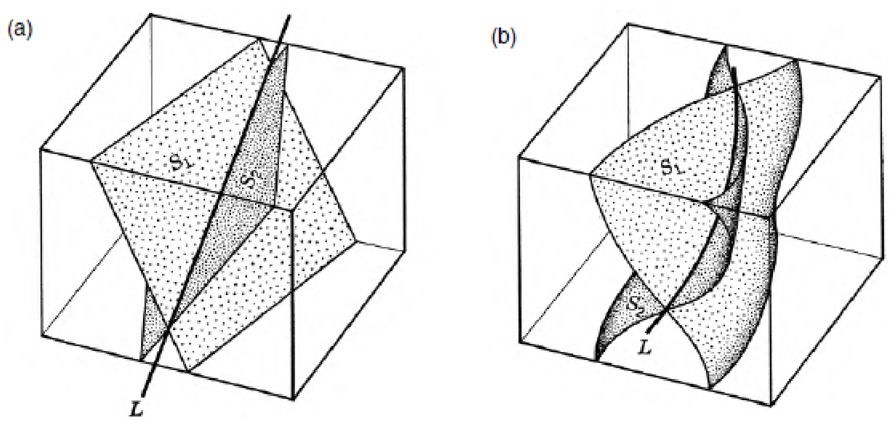

Fig 2.2: Discrete lineations defined by intersections of geological surfaces. (a) Straight lineation at intersection of two planar surfaces. (b) Curved lineation at intersection of curved surfaces.

If one could observe the entire fault surface or igneous contact surface, we suggest that the lineations would cover much if not all of these surfaces. This type of lineation is called a ***superficial lineation*** because it is only found on a discrete surface.

Where two roughly planar foliations with different attitudes exist in the same rock mass, say lithologic layering and a set of fractures, the mutual intersections define a penetrative lineation that permeates the rock mass . To the extent that both foliations are planar, the intersections define straight lineations.

PARAMETRIC REPRESENTATION OF CURVES

The spatial continuity of the set of points composing a curve is achieved by defining c as a continuous function. Because c is a vector quantity these functions are called vector functions. Curves in three-dimensional Euclidean space are defined in terms of vector functions of a single real variable, t, such that:

$$\mathbf{c}(t) = c_x(t)\mathbf{e}_x + c_y(t)\mathbf{e}_y + c_z(t)\mathbf{e}_z$$

The three scalar functions (c_x(t), c_y(t), c_y(t)) are the components of the vector function with respect to the base vectors (e_x, e_y, e_z). These functions, along with the base vectors, determine the position vectors for all points on the curve: as t varies smoothly from one value to another, the points trace out the curve. The vector equation written above is called the ***parametric representation of the curve***, and the real variable t is an arbitrary parameter for this representation.

Lineations are reported that lie on surfaces or within layers of sedimentary or metamorphic rock that approximate a cylindrical fold. Folds are termed cylindrical if a straight line moving parallel to itself can generate the surfaces of the surfaces of the layers. Straight line generator called ***fold-axis***.

UNIT TANGENT VECTOR

Local orientation of a curvilinear structure at the exposure is measured as the orientation of the line element that is tangential to the lineation.

consider the difference between the position vectors for two points on the curve, say s and s+▲.

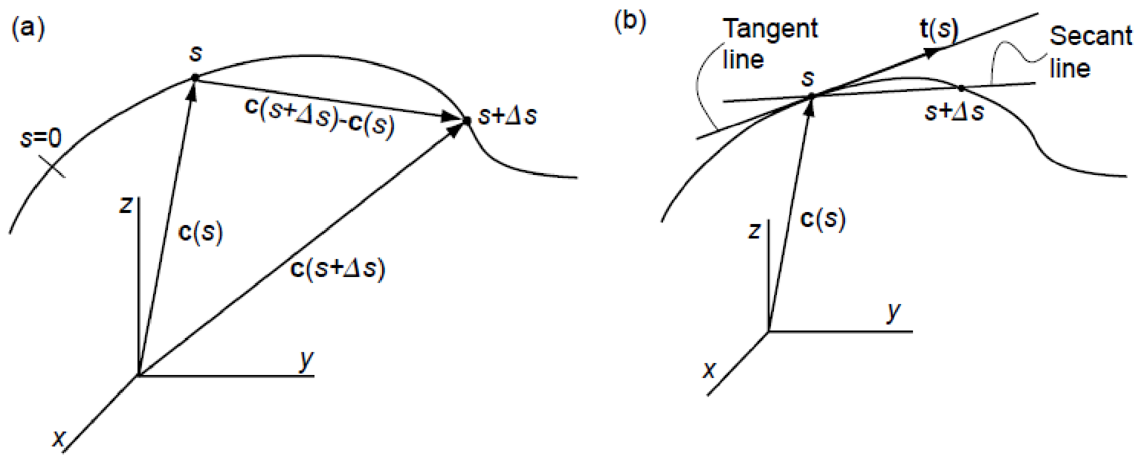

Fig 2.3: Diagrams to define unit tangent vector t. (a) Difference between two position vectors, c(s+▲s) and c(s), for curve defines vector parallel to the secant line. (b) In the limit, as the arc length, ▲s, goes to zero the secant line is parallel to the tangent line.

Dividing this difference by the arc length, ▲s, and taking the limit as this length goes to zero, we are left with the definition of the derivative of the vector function c with respect to the arc length▲s.

In the limit, as ▲s goes to zero the secant becomes parallel to the curve and of the same length as the arc. Therefore, this derivative is the unit tangent vector, t(s), at the point c(s).

$$\lim_{\Delta s \to 0} \frac{\mathbf{c}(s+\Delta s) - \mathbf{c}(s)}{\Delta s} = \frac{d\mathbf{c}}{ds} = \mathbf{t}(s)$$

THE CURVATURE VECTOR AND SCALAR CURVATURE

One of the two fundamental properties that uniquely determine the shape of the curve is curvature. The curvature vector, k, is defined for a natural representation of a curve, c(s), as the derivative of the unit tangent vector with respect to the natural parameter s.

$$\lim_{\Delta s \to 0} \frac{\mathbf{t}(s+\Delta s) - \mathbf{t}(s)}{\Delta s} = \frac{d\mathbf{t}}{ds} = \mathbf{k}(s)$$

The curvature vector is directed away from the curve on its concave side. The curvature vector points in the direction that the curve is turning.

The magnitude of the curvature vector , |k(s)|=k(s), a quantity known as the ***scalar curvature***.

The scalar curvature is equivalent to the spatial rate of change of the orientation of the unit tangent vector with arc length along the curve. The scalar curvature, k(s), is called an intrinsic property of a curve because it is one of two quantities that uniquely defines the shape of a curve. Where the orientation of the unit tangent vector changes more rapidly with position along the curve, the curvature is greater. A point on the curve where the curvature is zero is called an ***inflection point***.

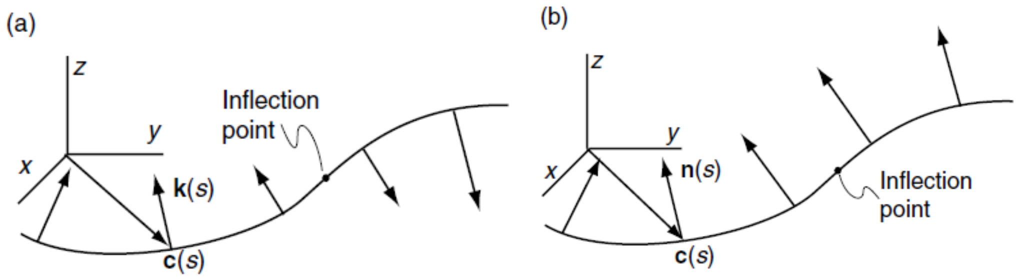

Fig 2.4: Diagrams to define unit principal normal vector, n. (a) Curve with curvature vectors, k(s). (b) Same curve with unit principal normal vectors, n(s).

THE UNIT PRINCIPAL NORMAL VECTOR AND BINORMAL VECTOR

In the interest of working with a geometric quantity that is less erratic in both magnitude and direction, a unit vector is defined as parallel to the curvature vector, but directed to remain continuous along the curve.

$$\mathbf{n}(s) = \frac{\pm\mathbf{k}(s)}{|\mathbf{k}(s)|}$$

The choice of sign in the numerator is used to keep this normal vector from switching direction arbitrarily from one side of the curve to the other at points of inflection.

In order to identify the second property that uniquely describes curves we again consider the natural parametric representation of a curve, c(s), and define a unit vector, b(s), called the ***unit binormal vector***, which is normal to the plane containing the unit tangent vector, t(s), and the unit principal normal vector, n(s):

$$\mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{n}(s)$$

The three unit vectors [t(s), n(s), b(s)] form the so called moving trihedron for the curve, which can be thought of as traveling along the curve with change in arc length, s.

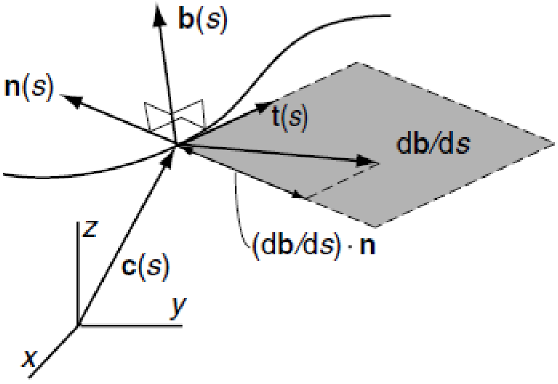

Fig 2.5: diagram showing trihedron for the curve.

THE SCALAR TORSION

The unit binormal vector is used to define the second intrinsic geometric property of curves, namely the torsion. It is defined as follows-

$$\tau(s) = -\left(\frac{d\mathbf{b}}{ds}\right) \cdot \mathbf{n}$$

The torsion is a measure of the change in orientation of the binormal vector, b, with arc length, s, but only that part of the change in orientation that projects onto the plane normal to the tangent vector, t. The torsion describes the component of rotation of the binormal vector about the tangent line with change in position along the curve. The torsion is called an intrinsic property of a curve because it serves, along with the scalar curvature, to define the shape of the curve uniquely.

To apply the concepts of curved lines from differential geometry to curvilinear structures observed at exposure, lineations must be sufficiently continuous so that the tangent, normal, and binormal vector functions and their first derivatives with respect to the natural parameter can be defined.

2.2 THE CONCEPT AND DESCRIPTION OF CURVED SURFACES

To make the appropriate deductions about the physical processes involved in the formation of faults, igneous contacts, unconformities, and other discrete geological surfaces one must quantitatively characterize the shapes of these surfaces.

PARAMETRIC REPRESENTATIONS OF CURVED SURFACES

A curved surface is defined as a continuous vector function of two scalar variables (u, v), called the parameters of the surface, such that s=s(u, v). The two parameters may be thought of as the coordinates of points on a plane, called the parameter plane, and those points map onto the surface according to the vector function s(u, v). As the two parameters vary, the heads of the successive position vectors sweep out the curved surface in three-dimensional space.

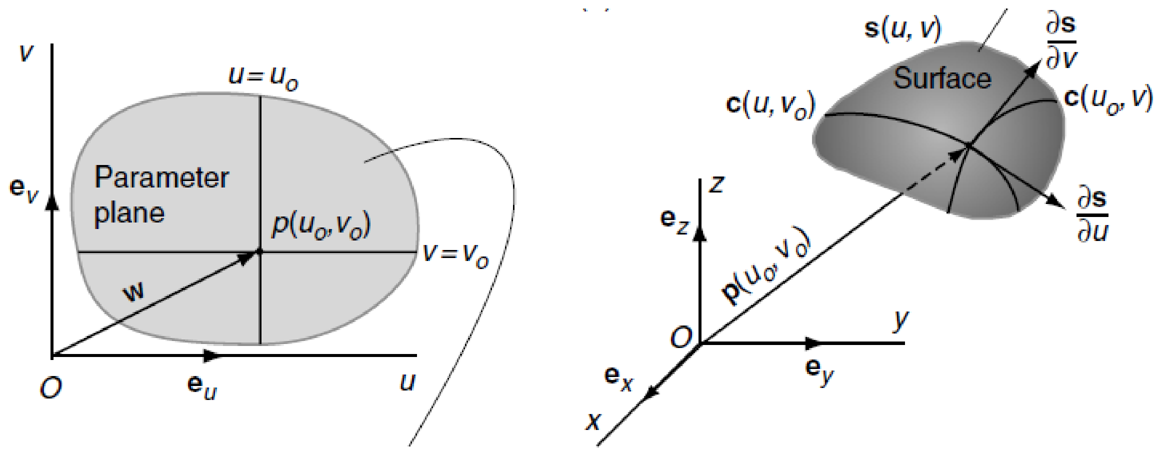

Fig 2.6: Parametric representation of a curved surface (a) Two-dimensional parameter plane with parameters u and v. Lines u=uo and v=vo in the parameter plane map to v- and u-parameter curves on the surface. (b) Three-dimensional surface defined by vector function of two parameters, s(u, v).

In many cases of interest in structural geology a simpler form of the parametric representation of a surface, or of a patch of a surface, may be found such that:

$$\mathrm{s}(u,v) = u\mathrm{e}_x + v\mathrm{e}_y + g(u,v)\mathrm{e}_z$$

THE TANGENT PLANE, TANGENT VECTOR, AND UNIT NORMAL VECTOR

Because the parametric representation of a surface, s(u,v), describes a vector function of two variable parameters, there is a partial derivative associated with each parameter. the partial derivatives of s(u,v) with respect to the two parameters are:

$$\frac{\partial \mathrm{s}(u,v)}{\partial u} = \frac{\partial s_x}{\partial u}\mathrm{e}_x + \frac{\partial s_y}{\partial u}\mathrm{e}_y + \frac{\partial s_z}{\partial u}\mathrm{e}_z$$
$$\frac{\partial \mathrm{s}(u,v)}{\partial v} = \frac{\partial s_x}{\partial v}\mathrm{e}_x + \frac{\partial s_y}{\partial v}\mathrm{e}_y + \frac{\partial s_z}{\partial v}\mathrm{e}_z$$

Thus, the partial derivative, ds/du, is a vector that is tangent to a u-parameter curve and points in the direction of increasing u. Similarly, ds/dv, is a vector that is tangent to a v-parameter curve and points in the direction of increasing v. These tangent vectors are not necessarily unit vectors.

The two partial derivatives of the vector function for a curved surface are used to define the parametric representation of planes, P, that are tangent to the surface. In general, the family of tangent planes for the surface, s(u, v), are defined as:

$$\mathrm{P} = \mathrm{s} + h\frac{\partial \mathrm{s}}{\partial u} + k\frac{\partial \mathrm{s}}{\partial v}, \quad -\infty \le h, k \le +\infty$$

The first term on the right-hand side is the position vector for the point on the curved surface. The second and third terms extend the position vector parallel to the tangent vectors at this point by arbitrary distances proportional to the variables h and k. As h and k range over the entire set of real numbers, this equation defines all possible
points on the tangent plane.

The tangent plane plays an important role in describing any curved surface. Furthermore, any geological surface observed at exposure is approximated locally with planar elements. These planar elements are tangent planes to the geological surface at the point of measurement.

The tangent vector, T, to the arbitrary curve is given by the derivative of the vector function c[u(t), v(t)] with respect to the parameter, t.

$$\begin{aligned}\mathrm{T} &= \frac{\mathrm{d}\mathrm{c}[u(t),v(t)]}{\mathrm{d}t} = \frac{\mathrm{d}\mathrm{c}(u,v_o)}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\mathrm{d}\mathrm{c}(u_o,v)}{\mathrm{d}v}\frac{\mathrm{d}v}{\mathrm{d}t} \\ &= \frac{\partial \mathrm{s}}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial \mathrm{s}}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}t}\end{aligned}$$

Because the tangent vector T is linearly dependent upon these two partial derivatives, it also lies in the tangent plane. In this way the tangent vector to the surface s(u,v) at an arbitrary point in any arbitrary direction is related to the partial derivatives of the parametric representation of the curved surface at that point.

The unit normal vector, N, makes a right-handed orthogonal system with the two tangent vectors, ds/du and ds/dv and is defined as:

$$\mathrm{N} = \frac{\frac{\partial \mathrm{s}}{\partial u} \times \frac{\partial \mathrm{s}}{\partial v}}{\left|\frac{\partial \mathrm{s}}{\partial u} \times \frac{\partial \mathrm{s}}{\partial v}\right|}$$

DIKE AND JOINT SURFACES IDEALIZED AS HELICOIDS

A dike in geology is a type of later vertical rock between older layers of rock. Technically, it is any geologic body which cuts across: flat wall rock structures, such as bedding; massive rock formations, usually igneous in origin.

Observations and mapping of opening fractures in rock, including basaltic dikes
and joints suggest that the surfaces of some of these fractures can be idealized as helicoids.

The parametric representation of helicoids is as follows:

$$\mathrm{s}(u,v) = (u\cos v)\mathrm{e}_x + (u\sin v)\mathrm{e}_y + (cv)\mathrm{e}_z$$

where v=constant (an angle) and u=constant (a length) are coordinate lines in the parameter plane that map onto the u- and v parameter curves on the helicoidal surface.

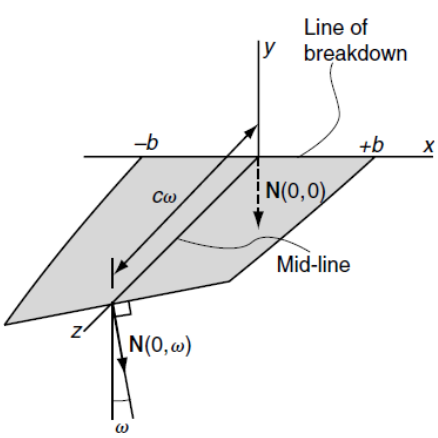

Fig. 2.7: The patch of a helicoidal surface used to model a fracture

<u>Definitions of some important mathematical terms:</u>

1. Twist of helicoidal surface- the angle between the reference unit normal, N(0, 0), and the unit normal at the point in question along the mid-line, N(0,v).

$$\mathrm{N}(0,0)\cdot\mathrm{N}(0,v)=[-(1)\mathrm{e}_y]\cdot[(\sin v)\mathrm{e}_x - (\cos v)\mathrm{e}_y]=\cos v$$

Therefore the twist angle is equal to the parameter v.

2. Spatial rate of twist- The component of s(0,v) in the z-direction is s_z=cv, and this is equal to the z-coordinate, so dz/dv=c. The spatial rate of twist of the surface is defined as dv/dz=1/c.

According to Fig. 2.7 the x-axis is equivalent to the line of breakdown from a single parent (main) fracture to multiple echelon fractures (feather fractures, twist hackle) in the fringe of a joint or dike.

By rigorous mathematical calculations, it was found that rate of twist is nearly constant(nearly). A constant rate of twist is consistent with the hypothesis that the hackle approximates a helicoidal surface.

Structural geologists quantify the orientation of a geological surface at an exposure by measuring strike and dip or dip and dip direction, and these measurements use the concept of the tangent plane at a point on a surface. The tangent vector to an arbitrary curve on the surface lies in the tangent plane. The normal vector to the curved surface is equivalent to the pole to a geological surface as plotted on a stereogram.

THE FIRST FUNDAMENTAL FORM, ARC LENGTH AND SURFACE AREA

A continuous curved surface is completely described at an arbitrary point in terms of two differential quantities called the first and second fundamental forms.

The fundamental forms are extremely important and useful in determining the metric properties of a surface, such as line element, area element, normal curvature, Gaussian curvature, and mean curvature.

The first fundamental form, I, at an arbitrary point on a curved surface, s(u,v), is a measure of the differential arc length of curves lying on the surface and oriented in all possible directions at that point. The points p(u,v) and p(u+du, v+dv),map onto the curved surface along the curve c[u(t), v(t)]. A tangent vector to this arbitrary curve is defined such that:

$$\mathrm{dc}=\mathrm{T}\mathrm{d}t=\frac{\partial \mathrm{s}}{\partial u}\mathrm{d}u+\frac{\partial \mathrm{s}}{\partial v}\mathrm{d}v$$

The first fundamental form, I, is a differential quantity defined as the scalar product of the differential tangent vector, dc, with itself-

$$I=\mathrm{dc}\cdot\mathrm{dc}=\left(\frac{\partial \mathrm{s}}{\partial u}\mathrm{d}u+\frac{\partial \mathrm{s}}{\partial v}\mathrm{d}v\right)\cdot\left(\frac{\partial \mathrm{s}}{\partial u}\mathrm{d}u+\frac{\partial \mathrm{s}}{\partial v}\mathrm{d}v\right)$$

$$=\left(\frac{\partial \mathrm{s}}{\partial u}\cdot\frac{\partial \mathrm{s}}{\partial u}\right)\mathrm{d}u^2+2\left(\frac{\partial \mathrm{s}}{\partial u}\cdot\frac{\partial \mathrm{s}}{\partial v}\right)\mathrm{d}u\mathrm{d}v+\left(\frac{\partial \mathrm{s}}{\partial v}\cdot\frac{\partial \mathrm{s}}{\partial v}\right)\mathrm{d}v^2$$

The coefficients in this equation are scalar quantities with particular geometric interpretations. Hence they are denoted with the special symbols E, F, and G. Using this notation the first fundamental form is written-

$$I=E\mathrm{d}u^2+2F\mathrm{d}u\mathrm{d}v+G\mathrm{d}v^2$$

$$E=\frac{\partial \mathrm{s}}{\partial u}\cdot\frac{\partial \mathrm{s}}{\partial u},\quad F=\frac{\partial \mathrm{s}}{\partial u}\cdot\frac{\partial \mathrm{s}}{\partial v},\quad G=\frac{\partial \mathrm{s}}{\partial v}\cdot\frac{\partial \mathrm{s}}{\partial v}$$

The scalar quantities E, F, and G are called the ***coefficients of the first fundamental form***. The values of the coefficients depend upon the choice of parameters used to represent the surface, but the first fundamental form itself is invariant with respect to this choice.
In this sense I is a property of the surface and plays a fundamental role in defining arc lengths on the surface.

The coefficients of the first fundamental form are scalar products of the tangent vectors to the u- and v-parameter curve.

The first fundamental form at a point on a surface, f(u,v), is the square of the differential arc length of a curve through that point.

The differential tangent vectors at the point p(u, v) in the directions of the u- and v-parameter curves are-

$$\mathrm{dc}(u,v_0) = \frac{\partial \mathrm{s}}{\partial u}\mathrm{d}u, \quad \mathrm{dc}(u_0,v) = \frac{\partial \mathrm{s}}{\partial v}\mathrm{d}v$$

These vectors form two sides of a small parallelogram and the differential area, dA, of this planar figure is used to approximate the area of the curved surface between the adjacent parameter curves.

$$\mathrm{d}A = |\mathrm{dc}(u,v_0) \times \mathrm{dc}(u_0,v)| = \left|\frac{\partial \mathrm{s}}{\partial u} \times \frac{\partial \mathrm{s}}{\partial v}\right| \mathrm{d}u\mathrm{d}v$$

$$= \left[\left(\frac{\partial \mathrm{s}}{\partial u} \times \frac{\partial \mathrm{s}}{\partial v}\right) \cdot \left(\frac{\partial \mathrm{s}}{\partial u} \times \frac{\partial \mathrm{s}}{\partial v}\right)\right]^{1/2} \mathrm{d}u\mathrm{d}v$$

$$\left(\frac{\partial \mathrm{s}}{\partial u} \times \frac{\partial \mathrm{s}}{\partial v}\right) \cdot \left(\frac{\partial \mathrm{s}}{\partial u} \times \frac{\partial \mathrm{s}}{\partial v}\right)$$
$$= \left(\frac{\partial \mathrm{s}}{\partial u} \cdot \frac{\partial \mathrm{s}}{\partial u}\right)\left(\frac{\partial \mathrm{s}}{\partial v} \cdot \frac{\partial \mathrm{s}}{\partial v}\right) - \left(\frac{\partial \mathrm{s}}{\partial u} \cdot \frac{\partial \mathrm{s}}{\partial v}\right)\left(\frac{\partial \mathrm{s}}{\partial u} \cdot \frac{\partial \mathrm{s}}{\partial v}\right)$$

Combining these results the differential area of the curved surface is $\mathrm{d}A = [EG - F^2]^{1/2}\mathrm{d}u\mathrm{d}v.$

The area of the helicoidal surface fracture calculated in the above manner is given as:

$$A = \omega\left[b\sqrt{b^2+c^2} + c^2 \ln\left(\frac{b+\sqrt{b^2+c^2}}{c}\right)\right]$$
$$A = \omega c^2\left[\frac{b}{c}\sqrt{\left(\frac{b}{c}\right)^2+1} + \ln\left(\frac{b}{c} + \sqrt{\left(\frac{b}{c}\right)^2+1}\right)\right]$$

This relationship demonstrates that the surface area of n helicoidal fractures, each of half-width b and length ωc, is less than the surface area of a single helicoidal fracture of half-width nb and length ωc. Taking the n fractures as a model for the twist hackle in the fringe region of a joint the non-intuitive result is that the surface area decreases as the number of fractures increases. On this figure each curve corresponds to a different twist angle, ω. For a twist angle of 1° (ω= Π/180) the surface area of ten fractures is 99.5% of that for the single fracture, only marginally less. However, for a twist angle of 30° (ω= Π /6) the surface area of ten fractures is 36.1% of that for the single fracture, dramatically less. Because the energy required to form a fracture in brittle materials scales with the fracture surface area his result shows that the breakdown of joints into hackle with helicoidal shapes is consistent with a condition of lesser energy expended during propagation.

THE SECOND FUNDAMENTAL FORM, SURFACE SHAPE, AND NORMAL CURVATURE

The second fundamental form provides a measure of the shape at any point on a continuous curved surface. Consider the arbitrary curve u= u(t), v=v(t) in the parameter plane which maps to the curve c[u(t), v(t)] on the curved surface. At an arbitrary point along this curve the differential tangent vector, dc, lies in the tangent plane to the surface . The unit vector, N, at this arbitrary point is a function of the two parameters u and v, such that the differential is:

$$dN = \frac{\partial N}{\partial u}du + \frac{\partial N}{\partial v}dv$$

The vector dN is a measure of the change in orientation of N with position along the curve on the surface and, in this sense, it is a measure of the shape of the surface. Also, because N is constant in magnitude, the vector dN is orthogonal to N and therefore lies in the tangent plane.

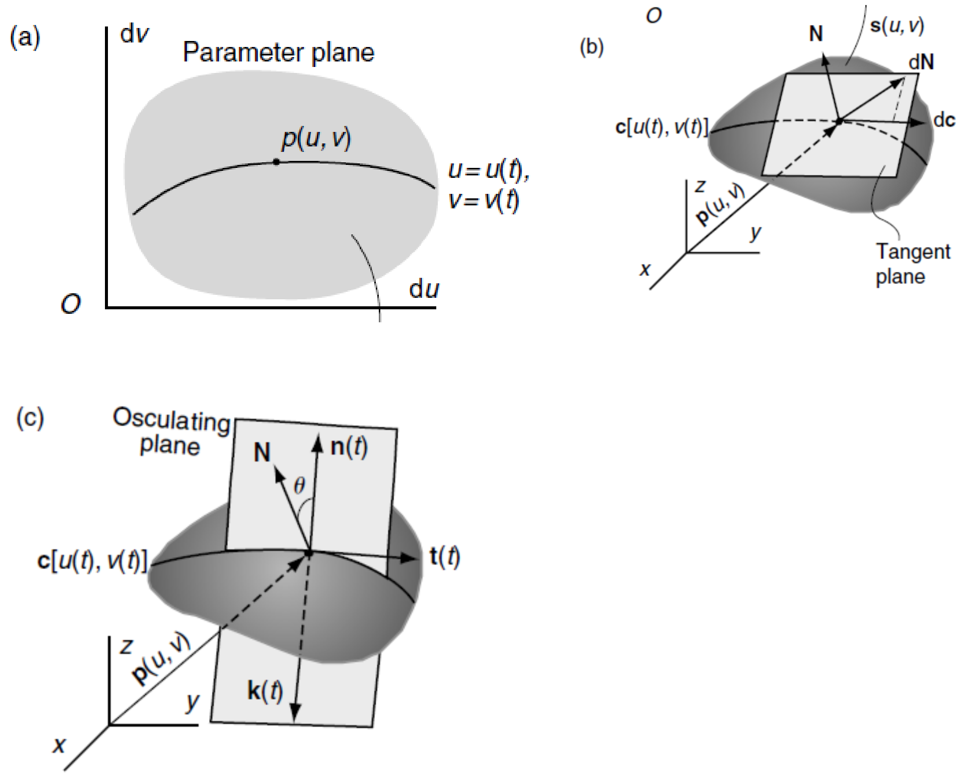

Fig 2.8: Diagrams to define second fundamental form for a surface. (a) Parameter plane with arbitrary curve u=u(t), v =v(t). (b) Surface with curve, c[u(t), v(t)], unit normal vector, N, differential tangent vector, dc, and differential normal vector, dN. (c) Surface with osculating plane containing the unit principal normal vector, n(t), and curvature vector, k(t).

Although dc and dN both lie in the tangent plane of the surface , these vectors are not necessarily parallel to one another. The shape of the surface in the particular direction specified by dc is characterized by the scalar product of the two vectors, dN and dc.

$$\begin{aligned} II = -dN \cdot dc &= -\left(\frac{\partial N}{\partial u}du + \frac{\partial N}{\partial v}dv\right) \cdot \left(\frac{\partial s}{\partial u}du + \frac{\partial s}{\partial v}dv\right) \\ &= -\left(\frac{\partial N}{\partial u} \cdot \frac{\partial s}{\partial u}\right)du^2 - \left(\frac{\partial N}{\partial u} \cdot \frac{\partial s}{\partial v} + \frac{\partial N}{\partial v} \cdot \frac{\partial s}{\partial u}\right)du\,dv \\ &- \left(\frac{\partial N}{\partial v} \cdot \frac{\partial s}{\partial v}\right)dv^2 \end{aligned}$$

The quantities L, M, and N are functions of the two parameters u and v, and are called the ***coefficients of the second fundamental form of the surface.***

$$II = Ldu^2 + 2Mdudv + Ndv^2$$

$$L = -\left(\frac{\partial \mathrm{N}}{\partial u} \cdot \frac{\partial \mathrm{s}}{\partial u}\right), \quad M = -\frac{1}{2}\left(\frac{\partial \mathrm{N}}{\partial u} \cdot \frac{\partial \mathrm{s}}{\partial v} + \frac{\partial \mathrm{N}}{\partial v} \cdot \frac{\partial \mathrm{s}}{\partial u}\right),$$

$$N = -\left(\frac{\partial \mathrm{N}}{\partial v} \cdot \frac{\partial \mathrm{s}}{\partial v}\right)$$

The coefficients depend upon the choice of parameters used to represent the surface, but the second fundamental form itself is invariant with respect to this choice , and in this sense II is a property of the surface. II characterizes the changing shape of the surface in all directions at a particular point and that the differential parameters du and dv define the direction.

The coefficients of the second fundamental form may be rewritten in a different way that is useful for computations.

$$L = \mathrm{N} \cdot \frac{\partial^2 \mathrm{s}}{\partial u^2}, \quad M = \mathrm{N} \cdot \frac{\partial^2 \mathrm{s}}{\partial u \partial v}, \quad N = \mathrm{N} \cdot \frac{\partial^2 \mathrm{s}}{\partial v^2}$$

One can use the coefficients of the second fundamental form to characterize the shape of a surface in the vicinity of a particular point as follows:

$$LN - M^2 \begin{cases} > 0, \text{ elliptic point} \\ = 0, \text{ parabolic point} \\ < 0, \text{ hyperbolic point} \end{cases}$$

$$L = M = N = 0, \ \text{planar point}$$

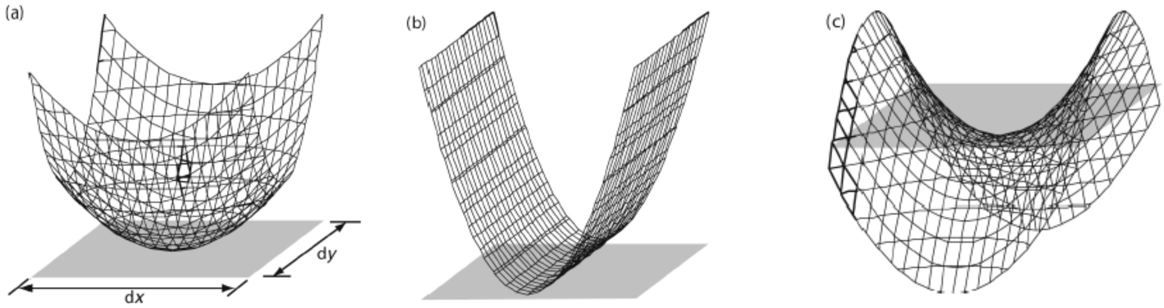

Fig 2.9: Three characteristic shapes of a surface near an arbitrary point: (a) elliptic, (b) parabolic and (c) hyperbolic.

The shape of a curved line is characterized, in part, using the curvature vector, k, and the scalar curvature, k=|k|. The osculating plane of a curve is the plane that contains both the unit tangent vector, t(t), and the unit principal normal vector, n(t).

The unit normal vector, N may not lie in the osculating plane of the curve. The normal curvature vector, k_n, and the normal curvature, κ_n, are defined in terms of k(t) and N(u,v) as:

$$\mathbf{k}_{\mathrm{n}} = (\mathbf{k} \cdot \mathrm{N})\mathrm{N}, \quad \kappa_{\mathrm{n}} = \mathbf{k} \cdot \mathrm{N}$$

The direction of the unit principal normal vector, n(t), is chosen for consistency along the curve. The normal curvature associated with a particular curve on a surface is equal to the curvature of this curve times the cosine of the angle between n and N. If the osculating plane of the curve contains the unit normal vector for the surface, then n and N are parallel, and $\kappa_n=\kappa$. On the other hand, if the osculating plane of the curve is parallel to the tangent plane for the surface, then $\kappa_n=0$. Hence it can be said that the curvature of an arbitrary curve at a point on a surface is greater than or equal to the normal curvature of the surface in the direction of the curve at that point.

CURVES ON SURFACES PROVIDE THE DIRECTION IN WHICH THE NORMAL CURVATURE OF THE SURFACE IS MEASURED, BUT THE NORMAL CURVATURE IS NOT NECESSARILY EQUAL TO THE CURVATURE OF THE CURVE.

The curvature, k, at a point on a curve is a unique property of the curve. The normal curvature, k_n, at a point on a curved surface varies in a smooth and systematic manner with the direction of the tangent line through the point of interest, from a maximum value, k_1, to a minimum value, k_2. These two values of normal curvature, k_1 and k_2, are called the ***principal normal curvatures***.

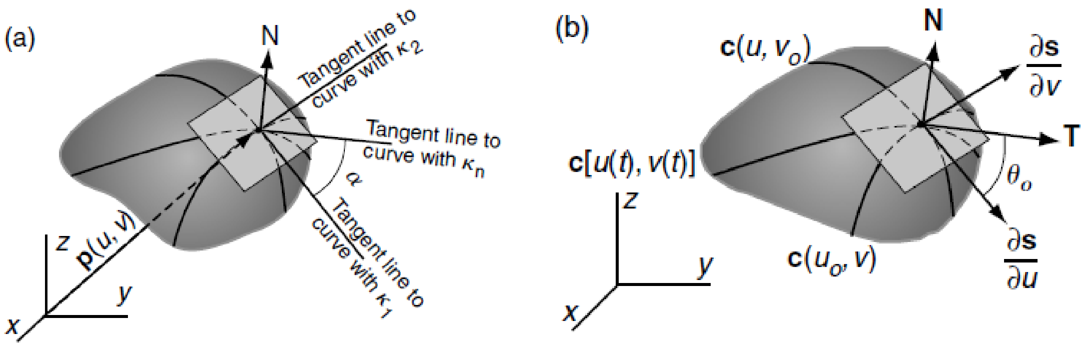

Fig 2.10: Diagrams to define variation of normal curvature with direction at a point on a surface. (a) Angle α measured in the tangent plane from direction of maximum principal normal curvature, k1, to direction of normal curvature kn. (b) Angle Θo measured in the tangent plane from tangent to u-parameter curve to direction of principal normal curvature.

The variation of normal curvature with direction is of the same form for all surfaces with continuous second partial derivatives such that:

$$\kappa_n = \kappa_1 \cos^2\alpha + \kappa_2 \sin^2\alpha$$

This relationship is known as ***Euler's Theorem*** and the angle α is measured in the tangent plane from the direction of the tangent line corresponding to the curvature k_1 to that corresponding to k_n.

The directions of the tangent lines associated with the extreme values of normal curvature are called the ***principal directions of normal curvature*** and they are orthogonal.
Because normal curvature is a property of surface at any point it can be written as a function of the fundamental forms. After working out a bit it is found as below-

$$\kappa_n = \frac{L(du/dt)^2 + 2M(du/dt)(dv/dt) + N(dv/dt)^2}{E(du/dt)^2 + 2F(du/dt)(dv/dt) + G(dv/dt)^2}$$

The normal curvature may be written in terms of the differentials, du and dv-

$$\kappa_n = \frac{L\,du^2 + 2M\,du\,dv + N\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2} = \frac{II}{I}$$

PRINCIPAL NORMAL CURVATURES, GAUSSIAN, AND MEAN CURVATURE

Euler's Theorem is used to calculate the normal curvature k_n in the direction of any line tangent to a surface, s(u,v), given the principal normal curvatures, k_1 and k_2, at a point on the surface. The method to calculate the magnitudes of the two principal normal curvatures and the principal directions is explained below-

From calculus we know that maximum or minimum values of a function in two variables are found by setting the partial derivatives of the function to zero.

$$\left.\frac{\partial \kappa_n}{\partial du}\right|_{(du_o, dv_o)} = 0, \quad \left.\frac{\partial \kappa_n}{\partial dv}\right|_{(du_o, dv_o)} = 0$$

When the resulting expressions are evaluated for the principal directions, du_0 and dv_0, the normal curvature takes on extreme values, k_0=**II/I**, satisfying the following linear equations-

$$(Ldu_0 + Mdv_0) + (Edu_0 + Fdv_0)(-\kappa_0) = 0$$
$$(Mdu_0 + Ndv_0) + (Fdu_0 + Gdv_0)(-\kappa_0) = 0 \qquad \ldots\ldots\ldots\ldots(2)$$

These equations have a simultaneous solution (1, $-k_0$) if the determinant made up of the coefficients on the left-hand side is zero. As we do so we get the following quadratic equation:

$$(LF - ME)(du_0)^2 + (LG - NE)du_0 dv_0 + (MG - NF)(dv_0)^2 = 0$$

The ratio of the direction numbers, dv_0: du_0=tanΘ_0, and this determines the angles, k_0 and k_0=Π/2, between the tangent to the u-parameter curve and the tangent to the principal directions. Arranging the terms further we get-

$$(MG - NF)\tan^2\theta_0 + (LG - NE)\tan\theta_0 + (LF - ME) = 0$$

The principal directions for the normal curvature at a point on a surface are found from the above equation using the standard formula for the solution of a quadratic equation. The magnitudes of the principal normal curvatures, k_1 and k_2, are found by rearranging (2) to factor out the two differentials du_0 and dv_0:

$$(L - \kappa_0 E)du_0 + (M - \kappa_0 F)dv_0 = 0$$
$$(M - \kappa_0 F)du_0 + (N - \kappa_0 G)dv_0 = 0$$

These two linear equations have a simultaneous solution (du_0, dv_0) if the determinant of the coefficients of the left side is zero. Expanding the determinant produces a quadratic equation in k_0:

$$(EG - F^2)\kappa_0^2 + (-EN + 2FM - GL)\kappa_0 + (LN - M^2) = 0$$

$$\kappa_0^2 - 2\kappa_m\kappa_0 + \kappa_g = 0$$

The first constant, k_m, is the average of the two principal normal curvatures and is referred to as the ***mean principal normal curvature***:

$$\kappa_m = \frac{(EN - 2FM + GL)}{2(EG - F^2)} = \frac{1}{2}(\kappa_1 + \kappa_2)$$

The second constant, k_g, is the product of the principal curvatures and is called the ***Gaussian curvature***:

$$\kappa_g = \frac{(LN - M^2)}{(EG - F^2)} = \kappa_1\kappa_2$$

The sign of k_g may be used to distinguish these three non-planar shapes. Taken together, the signs of the Gaussian and the mean curvature may be used to categorize surfaces with respect to orientation in keeping with geological conventions. This categorization provides a simple way to describe geological surfaces using the concepts of differential geometry.

PHASE 3

3.1 APPLICATION OF DIFFERENTIAL GEOMETRY TO STRUCTURAL GEOLOGY

This includes the examples which provide insights into how differential geometry can be applied to problems in structural geology.

CHARACTERISING THE SHAPES OF LINEATIONS ON DISCRETE SURFACES

Lineations are found on discrete geological surfaces such as faults and intrusive contacts.

On a fault, slickenlines should form a systematic pattern that reflects the relative motion of the two surfaces during frictional sliding. Of course the direction of relative motion at a point may change as a fault develops, leading to overprinting of slickenlines with different orientations. To illustrate the fact that slickenlines do vary systematically with position on a fault, and to appreciate some of the challenges inherent to the investigation of superficial lineations we turn to a data set from the Chimney Rock fault array. The four sets of faults in this region are displayed on the structure contour map constructed on the base of the Carmel Formation. Note, for example, that individual contours on this map are truncated by the Frenchman Fault. When traced to the north across the fault the sense of step is consistently to the east. However, the magnitude of the step decreases toward both terminations of this fault. This change in step magnitude suggests that the magnitude of the slip decreases from the mid-section of the Frenchman Fault toward the terminations and, indeed, the slip must go to zero at the terminations by definition.

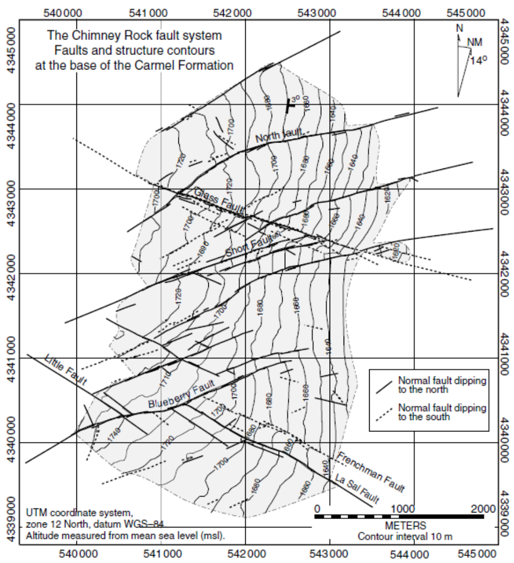

Fig 3.1: Structural contour map of blue-gray limestone unit near base of Carmel Formation illustrating offset by faults of the Chimney Rock array

Exposure of the faults at Chimney Rock are adequate to document the lateral variation in rake of the slickenlines over a distance of almost 3kms, but inadequate to document the vertical variation.

CHARACTERISING THE SHAPES OF FOLDED SURFACES

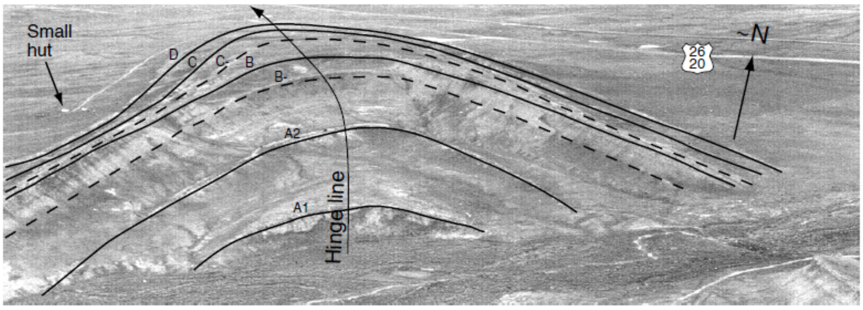

Fig 3.2: Oblique aerial photograph of the nose of the Emigrant Gap anticline, WY.

The Emigrant Gap anticline is a doubly plunging fold exposed for about 30 km along a northwest–southeast trend near Casper, Wyoming. Sandstone beds of the Frontier Formation crop out on the fold limbs and are continuously exposed around the fold hinge. The top of the lowest-most sandstone bed (labeled A1) of the Frontier Formation defines a somewhat asymmetric anticlinal surface, plunging gently to the north, with approximately planar limbs and a rounded hinge. The arc length of the exposed surface is on the order of 500 m, and the amplitude is about 75 m.

On a stereographic projection(fig 3.3), 543 poles to bedding define a great circle (labeled Π), with a tighter cluster of poles representing the eastern limb and a broader cluster representing the western limb. The pole to the great circle (labelled β) defines an approximate fold axis with trend and plunge of 345°, 04°.

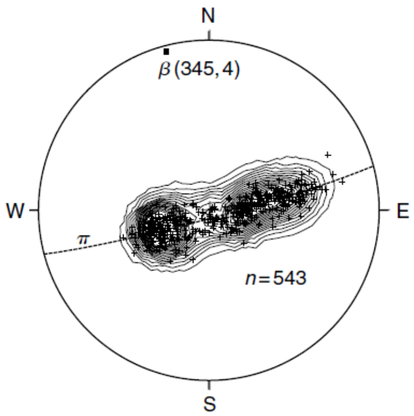

Fig 3.3: Stereographic projection of 543 poles to Frontier Formation sandstone beds on the Emigrant Gap anticline.

The GPS data were used to create a digital model of the surface, despite the limitation that the A1 sandstone is exposed only over about 25% of the map area. Clearly, sufficient data control is lacking to make conclusive statements about the geometry where the surface is not exposed. Eight cross sections of the folded surface were used to constrain the shape of the surface (Fig). From these cross sections one can infer that the anticline is asymmetric, with the west flank exhibiting steeper dips than the east flank, and the hinge is rounded.

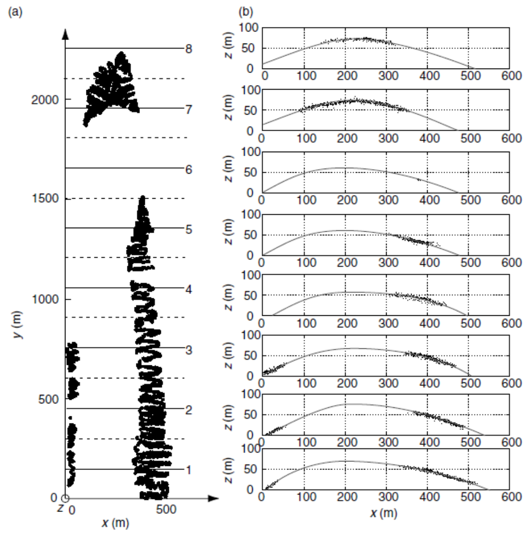

Fig 3.4: (a) Map view of Emigrant Gap anticline with 2529 GPS survey points . (b) Eight profiles of fold shape constrained by GPS data.

It is standard practice in structural geology to idealize folds as cylindrical structures. The principal normal curvatures provide a quantitative measure of the departure from a cylindrical shape. For a surface to be perfectly cylindrical one principal curvature must be zero everywhere, and the other principal curvature must have the same distribution on every cross section taken perpendicular to the fold axis.

Based on the signs of the Gaussian and mean curvatures at every grid point, the surface can be decomposed into areas that are locally shaped like one or the other of the six different characteristic shapes.

The characterization of folded surfaces using differential geometry will provide new insights concerning the process of folding.

FINAL CONCLUSION:

The structures encountered in Earth's crust are three dimensional with spatial variations in size and shape that only can be accounted for using a geometry that involves the spatial derivatives of such things as orientation and curvature. Plotting orientation data on a stereographic projection eliminates the opportunity to visualize and analyse these spatial changes. Furthermore, to proceed with modelling one needs to write boundary conditions that refer explicitly to geometry of surfaces. Finally, we now have precise field data on the three-dimensional shape of surfaces from new technology such as GPS and we need to know how to describe these surfaces and how to compare them to a model result.